

MULTIPLIERS : A GENERAL METHOD OF ANALYSIS FOR
CONSERVATION LAWS OF DIFFERENTIAL EQUATIONS

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Abstract

Conservation laws are studied using 'multipliers' - functions which produce divergences when they multiply an equation.

Multipliers are found for a number of well-known equations including those of interest in nonlinear physics such as the Korteweg-de Vries and Sine-Gordon equations. It is conjectured that multipliers exist for all conservation laws which are valid for all solutions of an equation.

The close links between multipliers and other properties of conservation laws are demonstrated and the identity - at least for Hamiltonian systems - of multipliers with the gradients of conservation laws is shown.

By using a formula for the variative of a product of two functions some previous results are obtained in a simple and direct way. It is found that the equation

$$u_t + u_n + R = 0, R \text{ polynomial,}$$

has at most one polynomial conservation law (the equation itself) unless n is odd.

The concepts of rank and irreducible terms used by Kruskal et al (J. Math. Phys. 11 952)) are generalised and are used to provide a completely new framework for the study of conservation

laws. This new framework is used to study the conservation laws of equations such as the Korteweg-de Vries equation and to generalise the result earlier obtained for $u_t + u_n + R = 0$. Recursion operators are studied and it is found that the concepts used in the framework can be used to give the general form that a recursion operator must take.

It is shown that the use of multipliers can produce results for systems of more than one equation by demonstrating that the known integrals for the Henon-Heiles system could be found using multipliers.

The framework developed can be incorporated in a computer program and a method of using multipliers by means of such a program is given and illustrated.

Introduction

Conservation laws are mathematical concepts which can be used to solve physical problems. A simple example is that of an elastic collision between two billiard balls. Without the use of conservation laws the solution is difficult; with conservation laws it is simple. If one uses the laws of conservation of momentum and energy there is only one possible solution and this solution can easily be found.

Conservation laws are closely related to other properties of a system and the study of the conservation laws of a system can reveal much about that system. Consider, for example, Noether's theorem. It states that, for a Lagrangian system, every symmetry has a related conservation law and this relationship can be used to find conservation laws. For example, if an equation does not have any explicit dependence on time then it will be invariant under a time-translation and, by Noether's theorem, it will have conservation of energy.

The close association of conservation laws with the other properties of an equation has recently become very apparent in the field of nonlinear evolution equations. This area has seen remarkable growth in the past twenty years. It had always been assumed that nonlinear systems could not have steady, stable solutions - it was assumed that such solutions would break down

due to the nonlinearities. In the late 1960's, it was found that the Korteweg-de Vries equation, $u_t + uu_x + u_{xxx} = 0$, had 'soliton' solutions which were remarkably and unexpectedly stable. An important property of the Korteweg-de Vries equation is the existence for the equation of an infinite number of independent conservation laws.

It was soon found that these properties of stable solutions and infinite sets of conservation laws existed in many physically useful equations. The Inverse Scattering Transform method of solution was discovered and could be used for these equations.

An exciting new area of research had been opened up and conservation laws played an important part in the initial discoveries and in the development of this area.

Given the importance of conservation laws and their close relationship with this new area of study, the solution of the problem

'What are the conservation laws of an equation?'

is of great interest. The work in this thesis began as an attempt to answer this question. The method that was used was to multiply an equation by a function (called a multiplier) and to try to rewrite the product as a divergence. The crux of the method was to find those multipliers which produced divergences and therefore conservation laws.

It was found that the use of multipliers was not restricted to finding conservation laws. It could also be used as a way of

studying other properties of the conservation laws of an equation, for example, recursion operators can be studied using multipliers.

The use of multipliers also produces results for conservation laws that were previously obtained by a variety of other methods. Each of these other methods worked for only a narrow range of equations and although a particular method may produce more results, or more comprehensive results, for a particular equation than the 'multiplier method' there can be few methods which produce results for such a wide range of equations as does the use of multipliers.

The use of multipliers also provides links between the results of previous workers which, at first sight, appear not to be directly related. For example, the work of Kruskal et al (1970) on the Korteweg-de Vries equation uses the concept of rank to study the conservation laws of the Korteweg-de Vries and Modified Korteweg-de Vries ($u_t + u^2 u_x + u_{xxx} = 0$) equations; a generalisation of this concept is shown to be directly related to the form of the recursion operators of the Korteweg-de Vries. One would expect such a relationship to exist in some way since, in both cases, it is the conservation laws of the Korteweg-de Vries equation that are being studied but the use of multipliers produces a direct and obvious relationship between the two.

Chapter one provides a broad outline of the general area of mathematical physics from which this work developed and surveys

in a little more detail the material that is considered necessary for an understanding of the importance and relevance of multipliers within the study of conservation laws, in particular those of evolution equations.

Chapter two discusses multipliers themselves. It gives examples of the previous uses of multipliers in part a) and in part b) it gives the theoretical links between multipliers and other properties of equations. Part c) gives a number of examples of multipliers for the conservation laws of some well-known equations and part d) reviews the work of my MSc thesis (Suttie (1981)) which provided the starting point for the work in this thesis.

Parts b) and c) of chapter two and chapters three, four and five are new and original work.

Chapter three uses a formula for the variational derivative of the product of two functions F and G to reproduce a number of previous results and to give a new result for the equation $u_t + u_n + R = 0$. This equation is shown to have at most one conservation law (the equation itself) unless n is odd. It is also shown that the multipliers for the equation $u_t + u_n + R = 0$ are also multipliers for the equation $u_t + u_x + u_n + R = 0$.

Chapter four introduces a framework for the study and use of multipliers. This framework generalises the ideas of rank and irreducibility of Kruskal et al (1970) to produce a completely new way of looking at conservation laws.

Chapter five uses the techniques of chapter four in the study of conservation laws and their properties.

CHAPTER ONE

Background

This chapter provides the necessary background for the work in this thesis. Part a) explains the notation and defines some of the basic concepts used. Part b) gives the general background to the work of this thesis - that is work which is related to this thesis but not immediately relevant to it while parts c), d) and e) give that background which is directly relevant.

a) Notation and Definitions

In this thesis the following notation is used:
(unless otherwise stated a subscript implies partial differentiation)

$$u_x = \frac{\partial u}{\partial x}$$

and the symbol

$$\partial_x \tag{1.1}$$

is used for the operator

$$\frac{\partial}{\partial x}.$$

A numerical subscript refers to differentiation by x and it is used only in two-dimensional systems with independent variables x

and t , thus

$$u_3 = \frac{\partial^3 u}{\partial x^3}$$

and $u_{2t} = \frac{\partial^3 u}{\partial x^2 \partial t}$.

(1.2)

If $A = A(x, t, u, u_x, \dots)$ then

$$A_x = \frac{\partial A}{\partial x}$$

$$A_2 = \frac{\partial^2 A}{\partial x^2}$$

$$A_{u_x} = \frac{\partial A}{\partial u_x}$$
(1.3)

and so on.

Summation over repeated indices is implied.

Due to the limitations of the word processor used to produce this work the symbol

≡

is used to represent

≡.

For an equation

$$G(t, x_1, \dots, x_n, u, u_t, u_{x_1}, \dots, u_{x_n}, \dots) = 0 \quad (1.4)$$

(this is usually abbreviated to $G(u) = 0$ or $G = 0$) a divergence

$$\frac{dT}{dt} + \frac{dX_1}{dx_1} + \dots + \frac{dX_n}{dx_n} \quad (1.5)$$

where $T = T(t, x_1, \dots, u, \dots)$ and $X = X(t, x_1, \dots, u, \dots)$, is called a conservation law if its value is zero for all solutions of $G=0$. (The shorthand $d_m J^m$ is often used for this divergence.)

T, X_1, \dots, X_n are functions of $t, x_i, m = 1 \dots n$, and of u and its derivatives. T is the conserved density and X_i is the flux in the i th direction.

A trivial conservation law is one for which T is equal to zero for all solutions of the equation or for which T is equal to an x -derivative. The term conservation law will not normally include trivial conservation laws.

If the fluxes are all zero on the boundaries then the conserved quantity Q is obtained from the conservation law by integration over the spatial variables.

$$\int \frac{dT}{dt} + \frac{dX_1}{dx_1} + \dots + \frac{dX_n}{dx_n} dx_1 \dots dx_n = 0$$

$$\Rightarrow \int \frac{dT}{dt} dx_1 \dots dx_n = 0$$
(1.6)

Defining $Q = \int T dx_1 \dots dx_n$, one gets

$$\frac{dQ}{dt} = 0$$
(1.7)

and Q is constant.

The conserved density, T , is the 'volume density' of Q .

If the fluxes are not zero on the boundaries, Q is no longer a constant. However the conservation law still holds at each point. In this case the fluxes on the boundaries can be regarded as measures of the 'amount of Q ' flowing into or out of the system.

Many workers deal with conserved densities instead of conservation laws. Since the existence of a conserved density

implies a conservation law and vice versa, the results are the same whichever approach is taken.

b) General Background

The work in this thesis has developed from the study of partial differential equations and in particular evolution equations with soliton solutions.

This area of study has grown tremendously in the past twenty years - the discovery of the Inverse Scattering Transform by Gardner et al (1967) and of an infinite number of conservation laws for the Korteweg-de Vries equation (Miura et al (1968)) could be regarded as starting points for this growth. Conservation laws have played an important role in this area.

In order to give the reader a feel for the environment out of which the work presented in this thesis grew, a wide-ranging outline of the relevant areas is given in this section. Subsequent sections develop those areas which are of particular importance to the work of this thesis.

- (i) Solitons [Scott et al (1973), Lamb (1980),
Eilenberger (1981), Ablowitz and Segur (1981),
Newell (1985)]

Until the late 1950's it had always been assumed that for dispersive non-linear systems, an initial solution would break down and the energy would be distributed amongst all the degrees of freedom of the system. Furthermore, it was assumed that even

if solitary waves were produced from very special initial conditions they would certainly break up on collision.

However a number of numerical experiments suggested that these assumptions were false. For example, Fermi, Pasta and Ulam (1955) studied a one-dimensional lattice (this led Kruskal and Zabusky (1965) to study the Korteweg-de Vries equation - a reduced form of the lattice equations) and Perring and Skyrme (1962) studied the Sine-Gordon equation as a model for elementary particles. It was found that the initial solutions were much more stable than expected and that solitary waves could survive collisions. The word 'soliton' was coined by Kruskal and Zabusky (1965).

The word soliton is often used imprecisely to mean any fairly stable solitary wave. Mathematically, it is restricted to solitary waves which retain their shape and which experience at most a phase shift upon collision with other solitons.

In a system which has strict mathematical soliton solutions, an initial profile evolves into a number of solitons plus an 'oscillatory tail'. In fact, a restricted set of initial conditions (Yoshinaga and Kakutani (1982)) is required to obtain a solution which does not have solitons.

An important property of solitons is that they are purely non-linear phenomena - it is not possible to obtain solitons by a perturbation of a linear system.

Thus the equations which have soliton solutions have

required an entirely new set of mathematical tools for their solution - some of these tools are outlined in this section.

The number of equations known to have soliton solutions is vast and growing and a large number of physical systems can be modelled by soliton equations or can be reduced to equations with soliton solutions.

For example, the following equations which have been intensively studied have soliton solutions;

the Korteweg-de Vries equation (KdV)

$$u_t + uu_x + u_{xxx} = 0, \quad (1.8)$$

the Sine-Gordon equation

$$u_{xt} = \sin u \quad (1.9)$$

and the nonlinear Schrodinger equation

$$iu_t + u_{xx} + 2u^2u^* = 0. \quad (1.10)$$

(ii) Inverse Scattering Transform [Lamb(1980), Ablowitz et al(1974), Ablowitz and Segur(1981)]

As its name suggests, the Inverse Scattering Transform uses scattering theory in the solution of non-linear differential equations. It can be regarded as a generalisation to non-linear equations of Fourier techniques.

Scattering is simply the interaction of a wave with some barrier - a potential. Transmission and reflection coefficients describe the result of the interaction. Inverse Scattering

consists of calculating the potential from the transmission and reflection coefficients.

Consider an equation in u and its derivatives

$$\begin{aligned} G(u(t)) &= 0, \\ u(0) &= G_0. \end{aligned} \tag{1.11}$$

In order to apply the Inverse Scattering Transform to this equation one must find an associated equation describing a scattering problem for which u is the (time-dependent) potential. The time-dependence of the potential is described by $G(u) = 0$. In the case of the Korteweg-de Vries (KdV) equation

$$u_t + u_x u + u_{xxx} = 0, \quad u \rightarrow 0 \text{ as } x \rightarrow \infty, \tag{1.12}$$

the associated equation is the Schrödinger equation

$$Z_{xx} - (u - s)Z = 0, \quad s \text{ is the eigenvalue corresponding to } Z. \tag{1.13}$$

If a suitable associated equation can be found, one can calculate the transmission and reflection coefficients at an initial time from the initial conditions on u . The time-dependence of the coefficients can then be found. At some later time the coefficients can be calculated using their known time-dependence and initial values. Inverse scattering theory then allows u to be calculated at this later time.

The method is best illustrated by example. The KdV equation was the first equation to be solved using the Inverse Scattering Transform and as it is one of the simplest its solution will be outlined to show the main principles of the method. [Details can be found in many texts and reviews - the original papers are

Gardner et al (1967,1974).]

One must first find the evolution of Z (and hence of the transmission and reflection coefficients) away from the potential (that is, where $u = 0$) and it is found that the equation has both a number of discrete eigenvalues ($s_n < 0$) and a continuous spectrum ($s > 0$). For $s_n < 0$

$$\begin{aligned} Z_n &= c_n(t) \exp(-K_n x) \text{ as } x \rightarrow \infty \\ K_n &= (-s_n)^{1/2} > 0. \end{aligned} \quad (1.14)$$

$$\Rightarrow c_n(t) = c_n(0) \exp(4K_n^3 t).$$

For $s = k^2 > 0$

$$\begin{aligned} Z &= \exp(-ikx) + b \exp(+ikx) \text{ as } x \rightarrow \infty \\ Z &= a \exp(-ikx) \text{ as } x \rightarrow -\infty \end{aligned} \quad (1.15)$$

$$\begin{aligned} \Rightarrow a(k,t) &= a(k,0) \\ b(k,t) &= b(k,0) \exp(8ik^3 t). \end{aligned}$$

Thus the asymptotic behaviour of Z has been determined for all t .

It is known from scattering theory that if

$$\begin{aligned} B(S) &= \frac{1}{2\pi} \int b(k) \exp(ikS) dk \\ &\quad + \sum_n c_n^2 \exp(-K_n S) \end{aligned} \quad (1.16)$$

then one obtains the Gel'fand-Levitan equation

$$K(x,y) + B(x+y) + \int_x K(x,z) B(y+z) dz = 0. \quad (1.17)$$

One then obtains the required solution to the KdV equation from

$$u(x,t) = -2 \frac{d}{dx} K(x,x). \quad (1.18)$$

For example, let $u(x,0) = -2 \operatorname{sech}^2 x$ (this solitary wave is a

soliton).

$$\text{From } Z_{xx} + (2\text{sech}^2 x + s)Z = 0 \quad (1.19)$$

one finds

$$K_1 = 1, c(0) = 2, b(k,0) = 0. \quad (1.20)$$

This is the reflectionless potential with only one discrete eigenvalue. The Gel'fand-Levitan equation is then

$$K(x,y) + 2 \exp(8t-x-y) + 2 \exp(8t-y) \int_x K(x,z) \exp(-z) dz = 0. \quad (1.21)$$

If $K(x,y) = L(x) \exp(-y)$

$$\text{then } L(x) + 2 \exp(8t-x) + 2 \exp(8t)L(x) \int_x \exp(-2z) dz = 0 \quad (1.22)$$

$$\Rightarrow L(x) = \frac{-2 \exp(x)}{1 + \exp(2x-8t)}$$

$$\text{and } K(x,y) = \frac{-2 \exp(x-y)}{1 + \exp(2x-8t)}$$

$$\Rightarrow u(x,t) = \frac{8 \exp(2x-8t)}{[1 + \exp(2x-8t)]^2} = -2 \text{sech}^2(x-4t). \quad (1.23)$$

Thus the initial profile maintains its shape.

$$\text{Similarly, if } u(x,0) = -6\text{sech}^2 x \quad (1.24)$$

then

$$b(k,0) = 0 \quad (1.25)$$

and there are two distinct eigenvalues

$$K_1 = 2, K_2 = 1. \quad (1.26)$$

One finally gets

$$u(x,t) = -12 \frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{[3\cosh(x-28t)+\cosh(3x-36t)]^2} \quad (1.27)$$

This has the asymptotic form of two widely separated solitons. Thus the initial profile breaks up into two stable solitons.

Unfortunately the calculations involved are not always so simple.

Since the solution of the KdV equation many other equations have been solved using the Inverse Scattering Transform. One particular scattering problem (called the AKNS after its discoverers, Ablowitz, Kaup, Newell and Segur (1974)) has proved to be associated with a wide range of physically important equations.

The scattering problem is

$$\begin{aligned} Z_{1x} + isZ_1 &= qZ_2 \\ Z_{2x} - isZ_2 &= rZ_1 \end{aligned} \quad (1.28)$$

(where Z_1 and Z_2 are two different variables).

If one assumes the time dependences

$$\begin{aligned} Z_{1t} &= AZ_1 + BZ_2 \\ Z_{2t} &= CZ_1 - AZ_2 \end{aligned} \quad (1.29)$$

(if D is used instead of $-A$ then it is found that $D = -A$)

then any coupled set of evolution equations in the variables q and r which obey the conditions

$$\begin{aligned} A_x &= qC - rB \\ B_x + 2isB &= q_t - 2Aq \\ C_x - 2isC &= r_t + 2Ar \end{aligned} \quad (1.30)$$

can be solved using the above scattering problem.

If one lets

$$\begin{aligned} A &= -4is^3 - 2iqr - qr_x + q_x \\ B &= 4qs^2 + 2iq_x s + 2q^2 r - q_{xx} \end{aligned} \quad (1.31)$$

$$C = 4rs^2 - 2ir_x s + 2qr^2 - r_{xx}$$

then, if $r = -1$, the Korteweg-de Vries equation in the form

$$q_t + 6qq_x + q_{xxx} = 0 \quad (1.32)$$

is obtained.

Among the other important equations which are associated with this scattering problem are the Modified Korteweg-de Vries (MKdV) equation

$$u_t + 6u^2u_x + u_{xxx} = 0, \quad (1.33)$$

the Sine-Gordon equation

$$u_{xt} = \sin u \quad (1.34)$$

and the Sinh-Gordon equation

$$u_{xt} = \sinh u. \quad (1.35)$$

A more general scattering problem that has been studied [Wadati et al (1979)] is

$$\begin{aligned} Z_{1x} + F(s)Z_1 &= G(s)qZ_2 \\ Z_{2x} - F(s)Z_2 &= G(s)rZ_1. \end{aligned} \quad (1.36)$$

For $F = is$, $G = s$, $r = -1$, $q = u^{-1}$ (and for appropriate A , B and C) one gets

$$u_t = 2(u^{-1/2})_{xxx}, \quad (1.37)$$

Dym's Equation [Kruskal (1975)].

The significance of the Inverse Scattering Transform lies, not only in its ability to solve specific problems such as those above, but also in the tremendous number of mathematical results which have been found by study of the method. One need only consult some of the introductory texts (for example, Ablowitz and

Segur (1981), Lamb (1980), Eilenberger (1981), Newell (1985) and the more advanced Bullough and Caudrey (1980)) to be in no doubt as to the importance of the Inverse Scattering Transform in the study of non-linear equations.

As one might expect from such a widely studied technique the Inverse Scattering Transform is open to a number of analytical interpretations.

Ablowitz and Segur (1981) show that it is

- a generalisation to certain nonlinear problems of the
Fourier transform,
- a canonical transformation to action-angle variables
of a completely integrable Hamiltonian system,
- a Backlund Transformation (see section iv).

(iii) Lax Representation [Lax (1968)]

The Inverse Scattering Transform method of solution was first found for the KdV equation. Its applicability to a wide range of equations became apparent when Lax (1968) was able to provide a representation that allowed generalisation to other equations.

Let L be a differential operator with a spectrum s , i.e.

$$LZ = sZ. \quad (1.38)$$

Differentiate this with respect to t ,

$$L_t Z + LZ_t = s_t Z + sZ_t \quad (1.39)$$

For Inverse Scattering Transform we require that $s_t = 0$.

If we represent Z_t by

$$Z_t = PZ \quad (1.40)$$

then we have

$$\begin{aligned} L_t Z + LPZ &= P(LZ), \text{ s constant with respect to } t, \\ &= (PL)Z \\ \Rightarrow L_t &= PL - LP \\ &= [P, L]. \end{aligned} \quad (1.41)$$

Thus if an equation can be cast into the form

$$L_t = [P, L], \quad (1.42)$$

then that equation would be susceptible to Inverse Scattering Transform using the scattering problem

$$LZ = sZ \quad (1.43)$$

and the time variation of Z given by

$$Z_t = PZ. \quad (1.44)$$

For example, let L be

$$L = -\frac{\partial^2}{\partial x^2} + u(x, t). \quad (1.45)$$

If $P = -\partial_x$ then

$$L_t = [P, L] \quad (1.46)$$

becomes

$$\begin{aligned} u_t &= -\partial_x u + u \partial_x \\ &= -u_x \end{aligned} \quad (1.47)$$

$$\Rightarrow u_t + u_x = 0.$$

This is perhaps the simplest wave equation and its solution is

$$u = f(x-t) \quad (1.48)$$

where f is any function. It represents a wave of constant shape and velocity propagating in the positive direction.

$$\text{If } P = -4 \frac{\partial^3}{\partial x^3} + 6u \frac{\partial}{\partial x} + 3u_x + f(t), \quad (1.49)$$

then the KdV equation is obtained.

Of course, the existence of a suitable scattering problem does not guarantee that the problem can be solved.

This representation can be generalised to matrices L and P . The AKNS problem described in the previous section can be put in the above form if (Gibbon (1985))

$$L = \left(-\frac{\partial^2}{\partial x^2} + V \right) \quad (1.50)$$

where

$$V = \begin{pmatrix} qr & q_x \\ r_x & qr \end{pmatrix} \quad (1.51)$$

and

$$P = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \quad (1.52)$$

The scattering problem is then

$$\left(-\frac{\partial^2}{\partial x^2} + V \right) Z = s^2 Z \quad (1.53)$$

where Z is a vector, $(Z_1, Z_2)^T$.

A more complex case is (Gibbon (1985))

$$L = \begin{bmatrix} 1/3 \partial_x & -1/3 A & -1/3 iB \\ 0 & -1/2 \partial_x & -1/2 A^* \\ i & 0 & \partial_x \end{bmatrix}, \quad (1.54)$$

$$P = \begin{bmatrix} is^2 & 2(sA - iA_x) & -2iA^*A \\ -2A^* & -is^2 & 2(iA_x^* - sA^*) \\ 0 & -2A & is^2 \end{bmatrix} \quad (1.55)$$

This produces the long-wave--short-wave resonance equations

$$\begin{aligned} iA_t - 2A_{xx} + 2AB &= 0 \\ B_t &= -4(A^*A)_x. \end{aligned} \quad (1.56)$$

N X N problems have also been studied. (See, for example, Moser (1975), Ablowitz and Haberman (1975) or the papers in Bullough and Caudrey (1980) by Wadati and by Calogero and Degasperis.)

(iv) Bäcklund Transformations [Miura (1976)]

A Bäcklund Transformation is a transformation between the solutions of one equation and those of another. If the two equations are the same then the transformation is often called an Auto-Bäcklund Transformation.

For example,

$$\begin{aligned} u_x &= -v - u^2 \\ u_t &= 6u^2u_x - u_{xxx} \end{aligned} \quad (1.57)$$

is a Bäcklund Transformation from the solutions of the KdV equation $[v(x,t)]$ to those of the Modified Korteweg-de Vries equation $[u(x,t)]$. The first equation is often referred to as the Miura Transformation between these two equations.

The system

$$\begin{aligned} \left(\frac{u+v}{2}\right)_x &= a \sin\left(\frac{u-v}{2}\right) \\ \left(\frac{u-v}{2}\right)_t &= \frac{1}{a} \sin\left(\frac{u+v}{2}\right) \end{aligned} \quad (1.58)$$

is an Auto-Bäcklund Transformation between two solutions $[u(x,t)$ and $v(x,t)]$ of the Sine-Gordon equation.

The scattering equations (1.28) are Bäcklund Transformations between themselves and the equation for which the scattering problem provides a solution. [Ablowitz and Segur (1981) p156].

If an equation has a Bäcklund Transformation then the Bäcklund Transformation can be used to derive new solutions from known ones. It is found that the Bäcklund Transformation for an equation such as the KdV, Modified Korteweg-de Vries or Sine-Gordon equations adds one soliton to the solution. It has been shown, Lamb (1974), that the solutions obtained in this way can have a simple relationship which enables new solutions to be easily calculated. Lamb shows that for the Sine-Gordon equation the following 'Theorem of Permutability' holds between four consecutive solutions, u_1, u_2, u_3 and u_4 ,

$$\tan\left(\frac{u_4 - u_1}{4}\right) = \left(\frac{a_1 + a_2}{a_1 - a_2}\right) \tan\left(\frac{u_2 - u_3}{4}\right) . \quad (1.59)$$

Bäcklund Transformations are closely related to the Inverse Scattering Transformation - the existence of one for an equation seems to imply the existence of the other. For the AKNS scattering problem Wadati et al (1975) have derived the Inverse Scattering Transform from a Bäcklund Transformation and vice

versa.

(v) Noether's Theorem [Noether(1918), Hill(1951), Sarlet and Cantrijn(1981), Rosen (1971,72,74)]

A Lagrangian system is one for which the equations of motion are given by

$$E(L) = 0 \quad (1.60)$$

where E is the Euler-Lagrange operator (see equation (1.72) below) and $L = L(x, t, u, u_x, u_t, \dots)$ is called the Lagrangian. For such a system Noether's Theorem states that for every transformation of a system that leaves the Lagrangian invariant there exists a conservation law.

This theorem provides an extremely useful method for finding conservation laws if the symmetry properties of the Lagrangian are already known.

The theorem also allows one to identify the conservation laws of a new equation. For example, the energy conservation law is associated with invariance under time-translation, momentum is associated with spatial invariance and angular momentum with rotational invariance.

One major problem with the standard formulation of Noether's Theorem is its restriction to Lagrangian systems. For symmetries of an equation which are not also symmetries of the Lagrangian or for symmetries of non-Lagrangian systems Noether's

Theorem has nothing to say.

In order to overcome this and other problems with Noether's Theorem a number of attempts have been made to generalise it.

Unfortunately, some of these generalisations have tended to obscure the relationship between symmetries and conservation laws. For example, Candotti, Palmieri and Vitale (1972) show that the entire set of symmetries is related to the entire set of conservation laws but the individual elements of the sets do not seem to be directly related. Although this result is not mathematically meaningless it is of no practical value if one wishes to calculate the symmetries or conservation laws of an equation.

One generalisation, due to Rosen (1974), allows an extension of Noether's Theorem to non-Lagrangian systems. (It does not, however, extend the symmetry-conservation law link.) Consider a transformation (y may be any of the independent variables t, x_1, \dots, x_n),

$$\begin{aligned} y &\rightarrow y' \\ u &\rightarrow u' \end{aligned} \tag{1.61}$$

$$\text{such that } y' = y + \delta y(y) \tag{1.62}$$

$$u'(y') = u(y) + \delta u(y)$$

$$\begin{aligned} \text{and } M(u) &= \delta u - \delta y u_y \\ &= u'(y) - u(y). \end{aligned} \tag{1.63}$$

Let $d_m J^m = 0$ be a conservation law associated with the above transformation for an equation $G = 0$.

Rosen's generalisation relates the transformation and the conservation law through the equation

$$M(u)G = d_m J^m + K \quad (1.64)$$

where K vanishes for solutions to $G = 0$ and is linearly independent of G . (In the terminology of this thesis $M(u)$ is a 'multiplier'.)

In the case of a Lagrangian system Rosen shows that this formulation is equivalent to the conventional formulation of Noether's Theorem since $K \equiv 0$ and the above equation applies whenever $M(u)$ is a symmetry of the Lagrangian.

Since Rosen's association of transformation and conservation law places no restriction on the properties of $M(u)$, the association is still valid for non-Lagrangian systems; that is the conservation law $d_m J^m = 0$ is associated with the transformation $M(u)$ which may or may not be a symmetry of the system. Thus for non-Lagrangian systems there is no direct symmetry - conservation law relationship through Rosen's generalisation of Noether's Theorem.

The value of Rosen's generalisation can be seen by considering a time translation,

$$\begin{aligned} \delta t &= C \\ \delta u &= 0 \\ \Rightarrow M(u) &= u_t \end{aligned} \quad (1.65)$$

Now consider the variation of the action integral $I = \int L \, dx$ for a Lagrangian under a transformation $(\delta u, \delta t)$

$$\delta I = \int \delta L \, dx. \quad (1.66)$$

The transformation is a symmetry of the Lagrangian if

$$\begin{aligned} \delta I &= 0 \\ \Rightarrow \delta L &= d_m J^m + K \end{aligned} \quad (1.67)$$

where $K = 0$ for solutions to the equations of motion.

If L is not explicitly dependent on time then, under a time translation

$$\begin{aligned} \delta L &= 0 \\ \Rightarrow d_m J^m &= 0. \end{aligned} \quad (1.68)$$

Rosen (1972) shows that if $\delta I = 0$ then, for the above transformation, the following conservation law is obtained

$$M(u)E(L) = d_m [P^m(L)u_t - L\delta_t^m] = 0 \quad (1.69)$$

where δ_t^m is the delta function

$$\begin{cases} = 1 & \text{if } m = t \\ = 0 & \text{if } m \neq t \end{cases} \quad (1.70)$$

$$P^m(L) = \sum_{a=0}^{\infty} \sum_{b=0}^a (-1)^b \frac{d}{dx_{m_1}} \cdots \frac{d}{dx_{m_b}} \left(\frac{\partial L}{\partial u_{m m_1 \dots m_a}} \right) \frac{d}{dx_{m_{b+1}}} \cdots \frac{d}{dx_{m_a}} \quad (1.71)$$

and E is the Euler-Lagrange operator,

$$E(L) = \sum_{a=0}^{\infty} (-1)^a \frac{d}{dx_{m_1}} \cdots \frac{d}{dx_{m_a}} \left(\frac{\partial L}{\partial u_{m_1 \dots m_a}} \right), \quad (1.72)$$

which produces the equations of motion

$$E(L) = 0. \quad (1.73)$$

$(m_1 \dots m_a)$ is a combination of m 's, for example,

$$xxt \rightarrow \frac{d}{dx_{m_1}} \cdots \frac{d}{dx_{m_3}} = \frac{d^2}{dx^2 dt} \quad (1.74)$$

Summation is implied over repeated indices.

Defining

$$T^{mt} = P^m(L)u_t - L\delta_t^m \quad (1.75)$$

one finds

$$M(u)E(L) = d_m T^{mt}. \quad (1.76)$$

For $M(u) = u_t$ it can be shown that $u_t E(L) = d_m T^{mt} = 0$ where T^{mn} is the energy-momentum tensor with T^{tt} as the energy and T^{mt} as the energy flux in the m th direction. The conservation law is then

$$d_m T^{mt} = 0. \quad (1.77)$$

Thus the multiplier u_t is associated with the conservation of energy via Rosen's generalisation.

For a non-Lagrangian system with the equation $G = 0$ the above analysis does not apply. Nevertheless it may be found that

$$u_t G = d_m J^m = 0 \quad (1.78)$$

is a conservation law for the equation $G = 0$. From the above example one can see that it is justifiable to regard the conservation law

$$d_m J^m = 0 \quad (1.79)$$

as the energy conservation law for this system. Thus Rosen's generalisation of Noether's Theorem allows an extension to non-Lagrangian systems.

An alternative formulation of the connection between conservation and symmetries is provided by Kumei (1977,78) using a Hamiltonian formulation. Kumei shows that for every conservation law of a Hamiltonian system there exists a one-

parameter group which leaves the Hamiltonian invariant and vice versa.

For example for the Sine-Gordon equation in the form [Kumei (1978)]

$$u_{tt} - u_{xx} + \sin u = 0, \quad (1.80)$$

and defining $q = u$ and $p = u_t$, the first four generators of invariance groups are

$$\begin{aligned} u_1 &= q_x \frac{\partial}{\partial q} + p_x \frac{\partial}{\partial p} \quad (\text{space invariance}) \\ u_2 &= p \frac{\partial}{\partial q} - (-q_{xx} + \sin q) \frac{\partial}{\partial p} \quad (\text{time invariance}) \\ u_3 &= (4q_{xxx} - 3q_x \cos q + 1/2 q_x^3 + 3/2 q_x p^2) \frac{\partial}{\partial q} \\ &\quad - (-4p_{xxx} + 3p_x \cos q - 3/2 q_x^2 p_x \\ &\quad \quad - 3/2 p_x p^2 - 3q_x q_{xx} p) \frac{\partial}{\partial p} \\ u_4 &= (4p_{xx} - p \cos q + 3/2 q_x^2 p + 1/2 p^3) \frac{\partial}{\partial q} \\ &\quad - (-4q_{xxx} + 5q_{xx} \cos q - 5/2 q_x^2 \sin q \\ &\quad \quad - \sin q \cos q - 3/2 q_x p^2 q_{xx} + 1/2 p^2 \sin q \\ &\quad \quad - 3q_x p_x p - 3/2 p^2 q_{xx}) \frac{\partial}{\partial p}. \end{aligned} \quad (1.81)$$

Corresponding to these four generators are the conserved densities

$$\begin{aligned} A_1 &= pq_x = \text{momentum density} \\ A_2 &= 1/2 p^2 + 1/2 q_x^2 - \cos q = \text{energy density} \\ A_3 &= 4pq_{xxx} - 3pq_x \cos q + 1/2 q_x^3 p + 1/2 q_x p^3 \\ A_4 &= -2p_x p^2 - 2q_{xx}^2 - 1/2 p^2 \cos q + 3/4 q_x^2 \\ &\quad + 1/8 p^4 - 5/2 q_x^2 \cos q + 1/2 \cos^2 q + 1/8 q_x^4. \end{aligned} \quad (1.82)$$

Tu (1982) shows that, for systems that have a Hamiltonian formulation, the two approaches, Noether's Theorem and group

theory, are equivalent.

A further formulation of the connection between symmetries and conservation laws has been developed by Guil Guerrero and Martínez Alonso (1980). They use algebraic methods instead of consideration of the variations of an action integral to show that conservation laws are associated with symmetries. They define a Lie-Bäcklund operator associated with a set of functions $M = (M^1 \dots M^k)$,

$$X_M = \sum_i D^i M^r \frac{\partial}{\partial u_i^r}, \quad (1.83)$$

where

$$D^i = \frac{d^i}{dt^{i_t} dx_1^{i_{x_1}} \dots dx_n^{i_{x_n}}}, \quad i = i_t + i_{x_1} + \dots i_{x_n}, \quad (1.84)$$

(u^r , $r = 1 \dots k$ are the dependent variables of the system).

$$\text{Using } \sum_i \frac{\partial F}{\partial u_i^r} D^i M^r = \sum_i D^i \left(\frac{\delta F}{\delta u_i^r} M^r \right) \quad (1.85)$$

where $\frac{\delta F}{\delta u}$ is the variational derivative [Galindo (1979b)] they get

$$X_M F = \sum_i D^i \left(M^r \frac{\delta F}{\delta u_i^r} \right). \quad (1.86)$$

X_M is an invariance operator for the system of equations $W^r[x, u] = 0$ if $X_M W^r \stackrel{\Omega}{=} 0$ where $\stackrel{\Omega}{=}$ implies equality for solutions of $W^r = 0$. This provides a symmetry of the equations of motion $W^r = 0$. X_M is called a Noether operator for a Lagrangian system if $\frac{\delta}{\delta u^r}(X_M L) = 0$, $r = 1 \dots k$, (that is, L is invariant under X_M up to a divergence).

Guil Guerrero and Martínez Alonso show that if X_M is a

Noether operator, then it is also an invariance operator for the equations

$$\frac{\delta L}{\delta u^r} = 0. \quad (1.87)$$

If $M^r \frac{\delta L}{\delta u^r}$ equals a divergence then

$$\frac{\delta}{\delta u^r} \left(M^s \frac{\delta L}{\delta u^s} \right) = 0 \quad \left(\frac{\delta L}{\delta u^s} = 0 \text{ are the equations of motion} \right) \quad (1.88)$$

$$\Rightarrow \frac{\delta X_{ML}}{\delta u^r} = 0$$

and vice versa.

Thus M provides a one-to-one correspondence between the conservation laws and Noether operators. It is also shown to provide a link between invariance operators and conservation laws for Hamiltonian systems.

The conclusion given above is very similar to Rosen's in that it relates conservation laws and symmetries without consideration of action integrals. An advantage of this is that the conclusions in no way depend on the boundary conditions of the system.

(vi) Conservation Laws and Symmetries

The existence of soliton solutions and of a scattering problem which solves an equation is linked with the existence of an infinite set of conservation laws. Although a formal mathematical link has not been established in general, it

certainly exists in particular cases and few would doubt that it will be found in the general case as well.

An example from Wadati et al (1975) of how the conservation laws can be derived from the Inverse Scattering Transform and from Backlund Transformations for the AKNS scattering problem illustrates the link.

The AKNS formalism uses the scattering problem

$$\begin{aligned} Z_{1x} - sZ_1 &= qZ_2 \\ Z_{2x} + sZ_2 &= rZ_1 \end{aligned} \quad (1.89)$$

with time dependency

$$\begin{aligned} Z_{1t} &= AZ_1 + BZ_2 \\ Z_{2t} &= CZ_1 - AZ_2. \end{aligned} \quad (1.90)$$

This leads to the conditions

$$\begin{aligned} A_x &= qC - rB \\ B_x - 2sB &= q_t - 2Aq \\ C_x + 2sC &= r_t + 2Ar. \end{aligned} \quad (1.91)$$

Wadati et al (1975) derive the conservation laws in the following way

$$\text{Let } P_1 = \frac{Z_2}{Z_1}, \quad P_2 = \frac{Z_1}{Z_2} \quad (1.92)$$

$$\begin{aligned} \text{Then } (qP_1)_t &= (A + BP_1)_x \\ (rP_2)_t &= (-A + CP_2)_x \end{aligned} \quad (1.93)$$

By expanding qP_1 as a power series

$$qP_1 = \sum_{n=1}^{\infty} f_n s^{-n} \quad (1.94)$$

and equating powers of s one obtains general formulae for the

infinite sets of conservation laws for the equations solvable by the above scattering problem.

This method produces all the polynomial laws for the Korteweg-de Vries equation [Konno, Sanuki and Ichikawa (1974)], the Modified Korteweg-de Vries equation [Wadati et al (1975)] and the Sine-Gordon equation [Sanuki and Konno (1974)]. Every second conservation law produced is trivial. The set of conservation laws produced for the Modified Korteweg-de Vries equation does not include the equation itself even though it is in conservation form.

Wadati et al (1975) also demonstrate a method of deriving conservation laws from Bäcklund Transformations. For example, the Bäcklund Transformation from the Korteweg-de Vries equation to itself is

$$\begin{aligned} w_x + w_x^1 &= -2s^2 + (w - w^1)^2/2 \\ w_t - w_t^1 &= -[2w_{xx} - 2w_x(w - w^1) \\ &\quad + 4s^2(w - w^1)]_x. \end{aligned} \quad (1.95)$$

Expanding $(w - w^1)$ in the following form

$$(w - w^1) = 2s + \sum_{n=1}^{\infty} f_n s^{-n} \quad (1.96)$$

and substituting into the above equations, a recurrence formula for f_n is obtained and the conservation laws are

$$(f_n)_t + (2qf_n + 4f_{n+2})_x = 0. \quad (1.97)$$

Haberman (1977) has a more general method which is closely related to the first method of Wadati et al and which also

involves infinite series.

The structure and properties of the infinite sets of conservation laws are discussed in section e).

As one would expect from Noether's Theorem these infinite sets of conservation laws are linked with infinite sets of symmetries.

Kumei (1978) shows that the KdV, Modified Korteweg-de Vries and Sine-Gordon equations all possess infinite sets of one-parameter symmetries. The symmetry generators given in (1.81) are the first four in the infinite set for the Sine-Gordon equation. These symmetries are quite separate from the Bäcklund Transformations as they are symmetries of the equation itself whereas a Bäcklund Transformation provides a relationship between different solutions of one equation or the solutions of different equations.

c) Hamiltonians

Hamiltonians were first developed in classical mechanics and are familiar to every physics student. This section reviews the theory of Hamiltonians and shows how the classical concepts have been generalised to the infinite-dimensional case.

A classical system with N degrees of freedom is Hamiltonian if it can be expressed in the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = - \frac{\partial H}{\partial q_i} \quad (1.98)$$

for $i = 1, \dots, N$ where H is any sufficiently differentiable function of p and q .

If one defines the Poisson bracket of two functions A, B as

$$\{A, B\} = \sum_{i=1}^N A_{q_i} B_{p_i} - A_{p_i} B_{q_i} \quad (1.99)$$

then $\{A, B\}$ has the following properties,

Antisymmetry: $\{A, B\} = -\{B, A\}$

Jacobi Identity: $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$ (1.100)

Leibnitz Rule: $\{A, BC\} = B\{A, C\} + C\{A, B\}$.

The last property is equivalent to saying that the Poisson bracket is a bilinear function.

Using this, one can see that

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial p_j} \frac{dp_j}{dt} + \frac{\partial A}{\partial q_j} \frac{dq_j}{dt} \\ &= - \frac{\partial A}{\partial p_j} \frac{\partial H}{\partial q_j} + \frac{\partial A}{\partial q_j} \frac{\partial H}{\partial p_j} \\ &= \{A, H\}. \end{aligned} \quad (1.101)$$

For a constant of motion, C , we have

$$\frac{dC}{dt} = \{C, H\} = 0. \quad (1.102)$$

Thus using the Jacobi identity one sees that the Poisson bracket of two constants of motion is itself a constant of motion.

Two constants of motion, C_1 and C_2 , are said to be 'in involution' if

$$\{C_1, C_2\} = 0. \quad (1.103)$$

A system with N degrees of freedom is said to be 'completely

integrable' if there exist N linearly independent constants of the motion that are in involution. A theorem of Liouville states that a completely integrable system is guaranteed to be solvable by quadrature.

In order to go over to the continuous case express (1.98) in the form

$$u_t = DgrH \quad (1.104)$$

where

$$\begin{aligned} u &= (q_1, \dots, q_N, p_1, \dots, p_N)^T \\ D &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ grH &= \left(\frac{\partial H}{\partial q_1} \dots \frac{\partial H}{\partial q_N}, \frac{\partial H}{\partial p_1} \dots \frac{\partial H}{\partial p_N} \right)^T. \end{aligned} \quad (1.105)$$

The Poisson bracket of A, B is then

$$\{A, B\} = (grA, DgrB) \quad (1.106)$$

$$\text{where } (A, B) = \sum_m A_m B_m.$$

This can be generalised if D is allowed to be any anti-symmetric operator, that is, $D^* = -D$ which is independent of u . It can be shown (Lax (1978)) that the Poisson bracket as defined has the three properties of antisymmetry, Jacobi identity and the Leibnitz rule. It can also be generalised to other symmetric operators which are dependent on u (see Calogero and Degasperis (1982)).

In the generalised case, one also gets

$$[F, H] = \frac{dF}{dt} \quad (1.107)$$

$$= [F, H] = 0 \text{ for constant } F.$$

Allowing N to go to infinity, A_j and B_j become functions, $A(u(x))$ and $B(u(x))$, (A, B) becomes

$$(A, B) = \int A(x)B(x) dx \quad (1.108)$$

and

$$\text{gr}A = \frac{\delta A}{\delta u} \quad (1.109)$$

where

$$\frac{\delta A}{\delta u} = \frac{\partial A}{\partial u} - \frac{d}{dx} \frac{\partial A}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial A}{\partial u_{xx}} \dots \quad (1.110)$$

The Poisson bracket retains the form used above (1.106).

A system of partial differential equations is said to be completely integrable if it has an infinite number of conservation laws in involution (that is, their Poisson bracket as defined above is zero). For example, the KdV equation is shown to be a completely integrable equation by Gardner (1971).

d) Integrating Operators and Bi-Hamiltonian Systems

Infinite dimensional Hamiltonian equations can be analysed using a method introduced by Magri (1978). This method uses potential operators to connect the conservation laws and symmetries of the system.

(i) Symmetries

If $u = u(x, t)$ and $K(u) = K(u, u_x, u_{xx}, u_{xxx}, \dots)$ then the

equation $u_t = K(u)$ is said to be an evolution equation if u vanishes on the boundary. The symmetries of this equation are those transformations

$$\bar{u} = u + \epsilon S(u) \quad (1.111)$$

where S is of the same form as K , which map every solution u into another solution.

Thus

$$\begin{aligned} \bar{u}_t - K(\bar{u}) &= u_t + \epsilon S(u)_t - K(u) - \epsilon K'_u S(u) \\ &= \epsilon [S'_u K(u) - K'_u S(u)] \end{aligned} \quad (1.112)$$

(S'_u is the Gateaux derivative of S defined by

$$S'_u v = \left. \frac{d}{d\epsilon} S(u + \epsilon v) \right|_{\epsilon=0}. \quad (1.113)$$

Thus an operator is a symmetry if and only if

$$S'_u K - K'_u S \stackrel{Q}{=} 0 \quad (1.114)$$

where $\stackrel{Q}{=}$ has the same meaning as in section b), part v.

From this it follows (see Magri (1978)) that the set of symmetry generators of

$$u_t - K = 0 \quad (1.115)$$

is a Lie-Algebra and

$$[S_j, S_k] \equiv S'_{ju} S_k - S'_{ku} S_j \quad (1.116)$$

is another generator.

(ii) Conservation Laws

A bilinear form for two functions a and b , $\langle a, b \rangle$ is defined as a bilinear map which associates a real number with the

elements a and b of two linear spaces A and B . The following bilinear form will be used in this section,

$$\langle a, b \rangle = \int a(x, t) b(x, t) dx. \quad (1.117)$$

Consider a function $Q(u)$. For an infinitesimal change, δu , in u , define the circulation as

$$\delta C \equiv \langle Q(u), \delta u \rangle. \quad (1.118)$$

It can be shown (Magri (1976)) that for two infinitesimal changes, du and δu , in u , if

$$\langle Q'_u du, \delta u \rangle = \langle Q'_u \delta u, du \rangle \quad (1.119)$$

(Q'_u is the Gateaux derivative of Q) for arbitrary δu and du then the function $Q(u)$ is conservative (that is, the circulation between two points depends only on the endpoints).

Such Q are called potential operators. For a potential operator one defines the potential $F(u)$ to be the circulation from some point u_0 to u . Since

$$\delta F = \langle Q, \delta u \rangle \quad (1.120)$$

Q is also called the gradient of F .

If $\frac{dF}{dt} = 0$ then a conservation law is said to exist for the equation $u_t - K = 0$. F is called an integral and Q is called an integrating operator.

From (1.120) it can be seen that

$$\begin{aligned} \frac{dF}{dt} &= \langle Q, u_t \rangle \\ &\stackrel{Q}{=} \langle Q, K \rangle. \end{aligned} \quad (1.121)$$

Thus if $\frac{dF}{dt} = 0$ then $\langle Q, K \rangle \stackrel{Q}{=} 0$ for all u .

(iii) Connecting Symmetries and Conservation Laws

A connection between the symmetries and conservation laws can be established if one can find an operator, L , such that

$$S = L_u Q. \quad (1.122)$$

If L_u obeys two conditions

$$\langle dv, L_u \delta v \rangle = -\langle \delta v, L_u dv \rangle \quad (1.123)$$

and

$$\begin{aligned} \langle dv, L'_u(\delta v; L_u \Delta v) \rangle + \langle \delta v, L'_u(\Delta v; L_u dv) \rangle \\ + \langle \Delta v, L'_u(dv; L_u \delta v) \rangle = 0 \end{aligned} \quad (1.124)$$

where L'_u is the Gateaux derivative of L_u with respect to u defined by

$$L'_u(v; w) = \left. \frac{d}{d\epsilon} (L_{u+\epsilon w}) \right|_{\epsilon=0}, \quad (1.125)$$

then it is said to be symplectic with respect to the bilinear form $\langle a, b \rangle$. For a potential operator, Q , a symplectic operator L_u linking Q with S by $S = L_u Q$ is called a Hamiltonian operator if and only if

$$\begin{aligned} \langle dv, S'_u L_u \delta v \rangle - \langle \delta v, S'_u L_u dv \rangle \\ = \langle dv, L'_u(\delta v; S(u)) \rangle. \end{aligned} \quad (1.126)$$

These three properties (1.123), (1.124) and (1.126) can be seen to be analogous to the three properties which define a Poisson bracket, that is antisymmetry, Jacobi identity and Leibnitz rule respectively.

For two Hamiltonian operators S_j, S_k the commutator $[S_j, S_k]$ is another Hamiltonian operator.

If F_j and F_k are functionals associated with S_j and S_k and Q_j is the operator given by $S_j = L_u Q_j$ then the following holds

$$\begin{aligned} \text{if } \{F_j, F_k\} &\equiv \langle Q_j, S_k \rangle = 0 \\ \text{then } [S_j, S_k] &= 0. \end{aligned} \quad (1.127)$$

Thus Q is an integrating operator and S is a symmetry operator for the equation

$$u_t = S \equiv L_u Q. \quad (1.128)$$

Thus if one can find L_u and Q such that $K(u) = L_u Q$, one has a connection between the conservation laws and the symmetries of the equation $u_t - K = 0$.

The form of the equation

$$u_t = L_u Q \quad (1.129)$$

is reminiscent of the Hamiltonian form

$$u_t = \text{Dgr}H \quad (1.130)$$

given by the previous section.

Thus one can say that L_u makes $u_t - K = 0$ Hamiltonian with Hamiltonian

$$\text{gr}H = Q. \quad (1.131)$$

Thus an operator L_u which makes an equation Hamiltonian maps its integrating operators onto its symmetry operators.

(iv) Bi-Hamiltonian Formalism

Important results can be easily obtained using the above analysis if an equation has two Hamiltonian decompositions. This can be seen using the KdV equation

$$u_t + uu_x + u_{xxx} = 0. \quad (1.132)$$

This can be decomposed into

$$u_t + \partial_x(1/2u^2 + u_{xx}) = 0 \quad (1.133)$$

$$\text{or} \quad u_t + (\partial_{xxx} + 2/3u\partial_x + 1/3u_x)u = 0. \quad (1.134)$$

Thus

$$L = \partial_x \quad (1.135)$$

and

$$M = (\partial_{xxx} + 2/3u\partial_x + 1/3u_x) \quad (1.136)$$

are two symplectic operators and

$$Q_1 = u \quad (1.137)$$

and

$$Q_2 = 1/2 u^2 + u_{xx} \quad (1.138)$$

are two potential operators.

Let S_2 be the symmetry operator associated with Q_1 ,

$$\begin{aligned} S_2 &= MQ_1 \\ &= uu_x + u_{xxx} \\ &= \partial_x Q_2 \\ &= LQ_2. \end{aligned} \quad (1.139)$$

Q_2 is obtained using the inverse of L . One can obtain a third symmetry by operating on Q_2 with M .

Thus

$$\begin{aligned} S_3 &= MQ_2 \\ &= \partial_x(5/18 u^3 + 5/3 uu_{xx} + 5/6 u_x^2 + u_{xxxx}) \end{aligned} \quad (1.140)$$

and

$$Q_3 = 5/18 u^3 + 5/3 uu_{xx} + 5/6 u_x^2 + u_{xxxx}. \quad (1.141)$$

In this way one can obtain an infinite sequence of integrating operators Q_n related by

$$\partial_x Q_{j+1} = (\partial_{xxx} + 2/3 u \partial_x + 1/3 u_x) Q_j. \quad (1.142)$$

This result was first discovered by Lenard (see Lax (1976)).

e) Conservation Laws

(i) Completely integrable equations

Definition

An equation is said to be completely integrable if it has an infinite number of conservation laws that are 'in involution'. Two conserved densities are in involution if their Poisson bracket is zero. For the KdV equation one can define the Poisson bracket as

$$\{A, B\} = \int \left(\frac{\partial A}{\partial u} \times \frac{d}{dx} \frac{\partial B}{\partial u} \right) dx. \quad (1.143)$$

For this Poisson bracket, one finds that if C_i and C_j are two conserved densities of the KdV equation then

$$\{C_i, C_j\} = 0 \quad (1.144)$$

for all i and j and thus the KdV equation is completely integrable.

The equations that appear in the study of solitons are often found to be completely integrable. The importance of complete integrability lies in the infinite-dimensional equivalent of

Liouville's Theorem - however, in the infinite-dimensional case the exact meaning of 'infinite' in 'an infinite number' must also be considered (Ablowitz and Segur (1981)).

Recursion Operators

Another important property of the infinite sets of conservation laws for completely integrable equations is that there often exists a recursion operator which produces the gradient of the $(n+1)$ th conserved density from the gradient of the n th.

Magri shows, using his operator formalism outlined in the previous section, how to construct recursion operators for Bi-Hamiltonian systems.

If M_1 is the first symplectic operator and M_2 is the second (higher order) operator then

$$M_1 Q_{j+1} = M_2 Q_j. \quad (1.145)$$

Thus for Dym's equation

$$u_t = (u^{-1/2})_{xxx} \quad (1.146)$$

one finds

$$M_1 v = 2uv_x + u_x v \quad (1.147)$$

and

$$M_2 v = v_{xxx}. \quad (1.148)$$

$$\text{Thus } (2u\partial_x + u_x)Q_{j+1} = \partial_{xxx}Q_j. \quad (1.149)$$

Painlevé Property

One of the most important questions in the area of completely integrable equations is

'Which equations are completely integrable?'

and related to it

'Given an equation, is it completely integrable?'

No complete answer has yet been found for these questions but there are many methods for finding the answer for particular equations or types of equations. One of these is the use of the 'Painlevé Property'.

Partial differential equations can often be reduced to ordinary differential equations for particular types of solution.

The KdV equation for example can be reduced to

$$U'' + 3U^2 - cU = K \quad (1.150)$$

for solutions of the form $u(x,t) = U(x-ct)$ or to

$$f''' + 6ff' = zf' + 2f, \quad (1.151)$$

for similarity solutions of the form

$$u(x,t) = (3t)^{-2/3} f(z) \quad (1.152)$$

where $z = x/(3t)^{1/3}$.

It has been found that every completely integrable equation can be reduced to an ordinary differential equation which has the Painlevé property (defined below) and, on the other hand, no equations which are not completely integrable have been found that can be similarly reduced.

The Painlevé property arises in the following way. An ordinary differential equation has singularities if its solution is infinite at some point. Such singularities can be classified as poles or as critical points. A critical point is said to be movable if its position depends on the constants of integration. If an ordinary differential equation has no movable critical points it is said to have the Painlevé property.

There are only fifty canonical second-order equations with this property including the so-called Painlevé Transcendents (the six equations of this set of fifty which were unknown before the work of Painlevé (see Ince (1927))).

(ii) Conservation Laws for other systems

Conservation laws exist for many equations not related to those discussed in section b) and many methods have been used to find them. They are often ad hoc methods which work only for a particular type of equation or conservation law. The results that follow indicate the type of results that have been found for a wide range of equations.

The conservation laws which are most familiar to physicists are those of classical mechanics. A well-known example is the Kepler problem of a body in a gravitational potential proportional to $1/r$. Since this a classical dynamical system energy, momentum and angular momentum are conserved. In addition,

there is another vector which is conserved - the Runge-Lenz vector

$$\underline{A} = \underline{p} \times \underline{L} - kM \underline{r}/r \quad (1.153)$$

where \underline{p} is momentum, \underline{L} is angular momentum, \underline{r} is position, M is mass and k is a constant. [See many advanced books on classical dynamics, for example, Saletan and Cromer (1971)].

The reader is no doubt familiar with many other similar conserved quantities for the ordinary differential equations of classical physics. The remaining conservation laws considered below are for partial differential equations.

Galindo (1981) studies linear evolution systems

$$\underline{u}_t = A(u) \quad (1.154)$$

where A is a matrix whose elements are linear differential operators or constants.

Many linear equations can be represented as evolution systems of this form. For example, the Classical Wave Equation

$$w_{tt} - w_{xx} = 0 \quad (1.155)$$

can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (1.156)$$

where $w = u$, $w_t = v$.

Galindo shows that if A meets certain conditions (which the Classical Wave Equation does not meet; see below) then the only conservation laws are at most quadratic in the field variables. This is also true for conservation laws which have explicit

dependence on the independent variables.

Well-known equations which meet the required conditions are the Klein-Gordon and Dirac equations.

Abellanas and Galindo [Galindo (1981), Abellanas and Galindo (1981c)] have shown that, for most linear equations, the conservation laws are at most quadratic in the dependent variable. There are exceptions however and one of the simplest is the Classical Wave equation

$$u_{tt} - u_{xx} = 0. \quad (1.157)$$

This can be written

$$(\partial_t - \partial_x)(u_t + u_x) = 0. \quad (1.158)$$

If $U = u_t + u_x$ then

$$U_t - U_x = 0. \quad (1.159)$$

This equation is known to have an infinite number of conservation laws - any function of U is a conserved density. Thus for the wave equation any function of $U = u_t + u_x$ is a conserved density. There may be other conservation laws which are not functions of $u_t + u_x$ (see chapter five).

One could use this procedure for other equations. For example the equation

$$u_{xxt} + u_{xt} + u_{xxx} + u_{xx} = 0 \quad (1.160)$$

can be written

$$(\partial_t + \partial_x)(u_{xx} + u_x) = 0. \quad (1.161)$$

Thus this equation also has an infinite number of conserved densities - every function of $u_{xx} + u_x$. Any equation which

factorises in this way has an infinite number of independent conservation laws of arbitrarily high order.

Abellanas and Galindo (1979, 81a, 83) have found conditions on non-linear equations

$$u_t + R = 0, \quad (1.162)$$

R a polynomial in u and its x -derivatives, for the existence of conservation laws. They define an equation to be even (odd) if its highest order derivative is even (odd).

They have shown that, for R of third order, the only non-linear equations with conservation laws of high order (thereby allowing the possibility of an infinite number of conservation laws) are the Korteweg-de Vries and Modified Korteweg-de Vries equations and equations of the form

$$u_t = u_{xxx} + (be^{au} - ce^{-au}) u_x - (a^2/8) u_x^3 + d u_x \quad (1.163)$$

where a, b, c, d are real [Abellanas and Galindo (1981a, 83)].

They have also found constraints on the form of fifth-order equations for the existence of high-order conservation laws.

In the case of even order equations they have found a number of criteria which limit the conservation laws of these equations [Abellanas and Galindo (1979)]. They show that the highest order derivative must appear linearly and hence the equations must be of the form

$$u_t = P = au_m + b \quad (1.164)$$

where a, b are functions in u and its x -derivatives up to order $(m-1)$. Let

$$P = A_x + B \quad (1.165)$$

$$\text{where } A = \int_0^{u_{m-1}} a \, du_{m-1} \quad (1.166)$$

$$B = P - A.$$

(For example if $u_t = u_6 u_4$

$$\text{then } A = \int_0^{u_5} u_4 \, du_5 = u_5 u_4 \quad (1.167)$$

$$B = u_6 u_4 - u_5 u_4.)$$

Let $c(\dots u_m)$ be a conserved density which is a polynomial in u and its x -derivatives up to the m th order. The criteria found by Abellanas and Galindo are

(1) If a only contains derivatives up to order $m-3$ and if

$$a \neq 0 \quad (1.168)$$

$$\text{and } b_{u_{m-1} u_{m-1}} (= \partial^2 / \partial u_{m-1}^2 b) \neq 0$$

then there exist no conserved densities which only contain derivatives up to order $(m/2)-2$.

For example,

$$u_t = u_6 + u_5^2$$

$$a = 1$$

$$b_{u_5 u_5} = 2 \quad (1.169)$$

$$(m/2)-2 = 1.$$

Thus there are no conserved densities which are functions only of u and u_x .

(2) If

$$\begin{aligned}
 & A_{u_{m-1}u_{m-1}} \neq 0 \\
 & \text{and } B_{u_{m-1}u_{m-1}} = 0
 \end{aligned}
 \tag{1.170}$$

then all conserved densities which only contain derivatives up to order $(m/2)-2$ are linear in u .

For example,

$$\begin{aligned}
 u_t &= u_5(u_6 + u_1) \\
 A &= \int_0^{u_5} u_5 du_5 = 1/2 u_5^2 \\
 B &= u_5 u_1 \\
 A_{u_5 u_5} &= 1 \\
 B_{u_5 u_5} &= 0
 \end{aligned}
 \tag{1.171}$$

$$(m/2-2) = 1.$$

Thus all $c(u, u_x) = ku$, k an arbitrary constant. There may, of course, be conserved densities containing higher derivatives.

(3) If

$$\begin{aligned}
 & A_{u_{m-1}u_{m-1}} = 0 \\
 & \text{and } B_{u_{m-1}u_{m-1}} \neq 0
 \end{aligned}
 \tag{1.172}$$

then there are no conserved densities containing only derivatives up to order $(m/2)-2$.

For example,

$$\begin{aligned}
 u_t &= u_6 u_4 + u_5^2 u \\
 A &= u_5 u_4 \\
 B &= u_5^2 u - u_5^2
 \end{aligned}$$

$$\begin{aligned}
 A &= 0 \\
 B &= 2u - 2.
 \end{aligned}
 \tag{1.173}$$

Thus there are no conserved densities which are polynomials only in u and u_x . Again, there may be conserved densities containing derivatives of higher order.

(4) Let $d(P)$ be the order in u of the highest order term in P .

$$\begin{aligned}
 \text{Then (a) If } d(B) \neq 0 \\
 \text{and } d(B) \neq d(A)
 \end{aligned}
 \tag{1.174}$$

then there exist no conserved densities containing only derivatives up to order $(m/2)-2$.

$$\text{(b) If } d(B) \neq d(A) + 1
 \tag{1.175}$$

then every conserved density contains only derivatives up to order $(m/2)-2$.

$$\begin{aligned}
 \text{(c) If } d(B) - d(A) \neq 0 \\
 \text{and } d(B) - d(A) \neq \pm 1,
 \end{aligned}
 \tag{1.176}$$

then there exist no conserved densities.

For example,

$$\begin{aligned}
 u_t &= u_4 + u_3^3 \\
 A &= u_3 \\
 B &= u_3^3 \\
 d(A) &= 1 \\
 d(B) &= 3 \\
 (m/2)-2 &= 0.
 \end{aligned}
 \tag{1.177}$$

From (b) the conserved densities are functions of u only and (a)

implies that there are no conserved densities which are functions of u only. Thus there exist no conserved densities for this equation.

Abellanas and Galindo (1982) also find conditions on $P(u...u_m)$ in the evolution equation

$$u_t = P \quad (1.178)$$

for the equation to be exceptional (that is, for it to have an infinite number of conservation laws and some simple symmetry properties). Some of these conditions are

1. m is odd,
 2. P is linear in u_m ,
 3. P does not depend on u_{m-1} ,
 4. dP/du_{m-3} is a divergence.
- (1.179)

For equations of the form

$$u_t = f(u...u_{2k+1}), \quad k = 1, 2, \dots \quad (1.180)$$

Tu (1980) found that for the equation to have four or more conservation laws it must be of the form

$$u_t = Cu_{2k+1} + f(u...u_{2k}), \quad C \text{ constant.} \quad (1.181)$$

Tu does not specify that k must be greater than or equal to one. However, the following equation (which is not of the required form) has an infinite number of independent conservation laws

$$u_t = u^n u_x, \quad n \text{ integer.} \quad (1.182)$$

Multiplying by u^m , m integer, produces the conservation law

$$\left(\frac{u^{m+1}}{m+1} \right)_t = \left(\frac{u^{m+n+1}}{m+n+1} \right)_x \quad (1.183)$$

Thus k must be greater than zero if Tu's condition is to apply.

This list of conservation laws gives an idea of the diversity that has been found amongst conservation laws.

CHAPTER TWO

Multipliers

The basic concept upon which the work in this thesis is based is that of a multiplier - a function which multiplies an equation in order to get a divergence. For solutions of the equation this divergence will equal zero and one will get a conservation law.

Multiplying an equation by some other function is a simple concept and it is not surprising that it has previously been used to obtain a number of results for differential equations. However, the significance of multipliers as a general method of studying conservation laws has not been recognised.

a) Previous uses of multipliers

In this section various previous uses of multipliers will be discussed.

The use of integrating factors to solve first-order ordinary differential equations is an example of the use of multipliers. Consider the equation

$$\frac{dy}{dx} + B(x)y = C(x). \quad (2.1)$$

Multiply this equation by an integrating factor $M(x)$ so that

$$M(x)\frac{dy}{dx} + M(x)B(x)y = M(x)C(x). \quad (2.2)$$

If the left-hand side can be integrated to give

$$\frac{d}{dx} (M(x)y) = M(x)C(x) \quad (2.3)$$

then the solution to the original equation is

$$y = \frac{1}{M(x)} \left[\int M(x)C(x) dx + \text{constant} \right]. \quad (2.4)$$

The integration can be done if

$$\begin{aligned} \frac{d}{dx} M(x) &= M(x)B(x) \\ \Rightarrow \frac{d}{dx} (\ln M(x)) &= B(x) \\ \Rightarrow M(x) &= e^{\int B(x) dx + \text{constant}}. \end{aligned} \quad (2.5)$$

The simplest conservation laws of an equation are usually found by using the multipliers u , u_x or u_t . For example, the first non-linear conservation law of the equation

$$u_{tt} + u_{xx} = 0 \quad (2.6)$$

is found by multiplying by u_t

$$\begin{aligned} u_t u_{tt} + u_t u_{xx} &= 0 \\ (1/2 u_t^2 - 1/2 u_x^2)_t + (u_x u_t)_x &= 0. \end{aligned} \quad (2.7)$$

Using Rosen's generalisation of Noether's Theorem we recognise this as the energy conservation law.

A simple extension of multiplication by u_t has been used by Sarlet, Bahar and others [Sarlet (1983), Sarlet and Bahar (1980,81), Ray and Reid (1980)] to find constants of the motion for second-order ordinary differential equations. A constant of the motion, C , is a conserved density for a conservation law in

one-dimension, that is

$$\frac{d}{dt} C = 0. \quad (2.8)$$

Noting that the multiplier u_t produces the energy equation, they consider a more general multiplier $g(t)u_t$ which they call an integrating factor.

They apply the integrating factor method to the equation

$$u_{tt} + V(t, u, u_t) = 0. \quad (2.9)$$

The time-dependent linear oscillator

$$u_{tt} + w^2(t)u = 0 \quad (2.10)$$

is of this form. Multiplication by $2r^2(t)u_t$ (where $r(t)$ satisfies the equation $r_{tt} + w^2r = Cr^{-3}$, C constant) produces the constant of the motion

$$Cr^{-2}u^2 + (ru_t - r_tu)^2. \quad (2.11)$$

Djukic and Sutela (1984) generalise this for the case of nonconservative dynamical systems.

Guil Guerrero and Martínez Alonso (1980) use multipliers in their study of Noether's Theorem. An outline of their results was given in chapter one, section b), part v. The function, M , is a multiplier for the equations

$$\frac{\delta L}{\delta u^r} = 0. \quad (2.12)$$

In his work on the generalisation of Noether's Theorem, Rosen (1974) uses multipliers to associate conservation laws with field variations. He considers the Dirac equations

$$F = (-ig^m \frac{d}{dx^m} - k)\bar{u} = 0$$

$$\bar{F} = (ig^m \frac{d}{dx^m} - k)u = 0,$$

where g^m are the Dirac matrices and k is constant.

The multipliers D and \bar{D} produce the divergence

$$FD + \bar{F}\bar{D} = d_m J^m$$

$$\text{if } D = (R(x) - f^m(x) \frac{d}{dx^m})u \quad (2.14)$$

$$\text{and } \bar{D} = (\bar{R}(x) - f^m(x) \frac{d}{dx^m})\bar{u}$$

where $\bar{R}(x) = g^0 R^+(x) g^0$, (R^+ is the adjoint of $R(x)$)

and, for $k = 0$,

$$\begin{aligned} f^m(x) &= a^{mn}x_n + b^m + rx^m + c_n(x^m x^n - 1/2 g^{mn} x^2) \\ R(x) &= (a_{mn} + c_n x_m - c_m x_n)^{1/4} [g^m, g^n] \\ &\quad - 3/2 (r + c_n x^n) + iA + iBg^5, \end{aligned} \quad (2.15)$$

or for $k \neq 0$

$$\begin{aligned} f^m(x) &= a^{mn}x_n + b^m \\ R(x) &= 1/4 a_{mn} [g^m, g^n] + iA \end{aligned} \quad (2.16)$$

($[g^m, g^n] = g^m g^n - g^n g^m$ is the commutator; a^{mn} , b^m , c_m , r , A , B constants).

Rosen's aim in this work is to associate conservation laws with transformations of the fields. In this case the various constants are associated with the following transformations

- a^{mn} - Lorentz transformations
- b^m - space-time translations
- c_m - conformal transformations
- r - dilation transformations

A - phase transformations

B - chiral transformations.

Rosen's work studies multipliers as transformations. Thus multipliers such as those for the Benjamin-Ono equation, (used in the study of long deep-water internal waves [Ablowitz and Segur (1981)])

$$q_t + 2qq_x + H(q_{xx}) = 0 \quad (2.17)$$

where $H(q(z)) = \int \frac{q(z)}{z-x} dz$, (the integral is the usual principal value), (2.18)

are not seen to be related to the multipliers of the simpler equations that we have been considering. For example, the second multiplier (see section c) of the Benjamin-Ono equation is

$$3q^2 + 3/2(H(q_x)) - 3/2q_x H. \quad (2.19)$$

The third term operates on the equation itself and cannot be directly related to a transformation by Rosen's method.

A generalised form of multiplier, an integro-differential operator, has been used by McGuinness (1978,1980a,b) to show that the conservation laws of equations which are solvable by the Inverse Scattering Transform with the scattering problem of Ablowitz et al (1974) can be regarded as the energies of a hierarchy of equations of increasing order. For example, for the Korteweg-de Vries equation in the form

$$u_{xt} + u_{xx}u_x + u_{xxxx} = 0 \quad (2.20)$$

the conservation laws are of the form

$$u_t H^n(u_{xt} + u_{xx}u_x + u_{xxxx}) = d_m J^m = 0, \quad n = 0, 1, 2, \dots \quad (2.21)$$

where

$$H = \frac{d^2}{dx^2} + 2/3 u_x + 1/3 u_{xx} \int_{-\infty}^x dx. \quad (2.22)$$

Since by Rosen's generalisation of Noether's Theorem multiplication by u_t produces an energy equation, the conservation laws of the Korteweg-de Vries equation are the energy equations of the hierarchy of equations

$$H^n(u_{xt} + u_{xx}u_x + u_{xxxx}) = 0. \quad (2.23)$$

b) Theoretical considerations

In part (i) the existence of multipliers is discussed and in part (ii) multipliers are shown to have direct links with other properties of integrable equations.

(i) Existence of multipliers

An important question regarding multipliers is: how widespread are multipliers? Do multipliers exist for all, some or hardly any conservation laws?

My conjecture is that multipliers exist for all conservation laws which are valid for all solutions of an equation. Every such conservation law or constant of the motion that I have tested has a multiplier. The great variety of equations studied in this chapter supports this conjecture.

If a conservation law is not valid for all solutions then it may not have a multiplier. For example, Miura et al (1968) find

extra conservation laws for periodic solutions of the KdV while Lin and Chen (1982) find conservation laws for the Kadomtsev - Petviashvili equation

$$u_{xt} + u_{xx}u_x + u_{xxxx} + u_{yy} = 0 \quad (2.24)$$

for solutions which tend to zero on the boundaries.

In such cases one is looking at conservation laws of an equation plus constraints and new conservation laws might arise in the following way. If one takes an equation $G(u) = 0$ and multiplies it by some function $F(u)$ one gets

$$FG = \frac{d}{dt} T + \sum_1 \frac{d}{dx_1} X_1 + K. \quad (2.25)$$

If $K = 0$ is a constraint on $G = 0$ then, for solutions of $G = 0$ which also obey $K = 0$ one gets the conservation law

$$FG = \frac{d}{dt} T + \sum_1 \frac{d}{dx_1} X_1. \quad (2.26)$$

In this case $F(u)$ is not a multiplier since this conservation law is not valid in general - that is FG cannot be written as a divergence unless u is a solution of $K = 0$. Thus a system consisting of an equation plus constraints can have conservation laws which are not valid for all solutions of the equation itself.

Any conservation law

$$d_m J^m = 0 \quad (2.27)$$

which is valid for all solutions of $G = 0$ can be written

$$d_m J^m = FG + L \quad (2.28)$$

where $L = 0$ when $G = 0$. Every solution of $G = 0$ is a solution of

$L = 0$ but there could be solutions of $L = 0$ which are not solutions of $G = 0$. This is obviously the case if $L = F'G$ since $L = 0$ will have solutions if $G = 0$ or if $F' = 0$. This means that $L = 0$ is not a constraint on $G = 0$ since it does not impose any restriction on the solutions of $G = 0$.

I know of no case where $L \neq F'G$ and it seems unlikely that this could ever be the case. Hence my conjecture that multipliers exist for all conservation laws which are valid for all solutions of an equation.

This conjecture has not yet been proved for all conservation laws for all equations. It can, however, be proved in certain circumstances. For example, Martínez Alonso (1979) proves it for the case of 'normal' systems, that is systems of the form

$$\frac{\partial^n u}{\partial t^n} = P(x, t, u) \quad (2.29)$$

where P is a function of x , t , u , the x -derivatives of u and the t -derivatives of u up to order $(n-1)$.

Djukic and Sutela (1984) prove a similar result for the first integrals of nonconservative dynamical systems. They show that for every constant of motion there is an integrating factor (that is, a multiplier).

They also state that this result is equivalent to the inverse Noether's Theorem. Thus the existence of multipliers is linked with Noether's Theorem. This agrees with Rosen's results outlined in chapter one, section b, part v.

(ii) Links with other work

The work of Magri (chapter one, section d) enables us to provide a link between multipliers and the other properties of completely integrable equations. In chapter one it was shown that for a completely integrable evolution equation, $u_t - K = 0$, and a conserved quantity for that equation, C ,

$$\begin{aligned}\frac{dC}{dt} &\equiv \langle Q, u_t \rangle \\ &= \langle Q, K \rangle\end{aligned}\tag{2.30}$$

where Q is a potential operator and the bilinear form of chapter one, section d is used.

If a multiplier, M , exists for an evolution equation then

$$Mu_t = \frac{dT}{dt}\tag{2.31}$$

where T is the conserved density of the conservation law. The conserved quantity associated with M is

$$\begin{aligned}C &\equiv \int T dx \\ \Rightarrow \frac{dC}{dt} &= \int \frac{d}{dt} T dx \\ &= \int Mu_t dx \\ &= \langle M, u_t \rangle.\end{aligned}\tag{2.32}$$

Thus M can be identified as a potential operator for the evolution equation and therefore it is the gradient of the conserved density.

If Q is a potential operator then

$$\frac{dC}{dt} = \langle Q, u_t \rangle\tag{2.33}$$

and if C can be written

$$C = \int T \, dx \quad (2.34)$$

then

$$\begin{aligned} \int Q u_t \, dx &= \int \frac{dT}{dx} \, dx \\ \Rightarrow Q u_t &= \frac{dT}{dt} + \frac{dX}{dx}. \end{aligned} \quad (2.35)$$

If QK can be written as an x -derivative $\frac{dX'}{dx}$ then

$$Q(u_t - K) = \frac{dT}{dt} + \frac{d(X - X')}{dx} \quad (2.36)$$

$= 0$, for solutions to the equation.

However, equation (2.36) does not guarantee that $Q u_t$ can be written as a divergence, only that its value is the same at any point as a divergence. Thus one may have

$$Q(u_t - K) = T_t + X_x + K, \quad (2.37)$$

where K is zero for solutions to the equation. This is the same form as equation (2.25).

Thus the existence of a potential operator for a conserved quantity does not guarantee the existence of a multiplier, but if a multiplier exists it will be a potential operator and therefore the gradient of the conserved density.

Since potential operators can exist for non-Hamiltonian systems it can be seen that the equation need not be Hamiltonian for multipliers to exist. However, if the system is not Hamiltonian then the relationship between symmetries and conservation laws that is provided by potential operators will not exist.

Thus Magri's work mathematically relates multipliers to the other properties of the equation.

From what has been shown above, multipliers can be seen to be related to the work of a number of other workers.

Kumei (1977) derives a result similar to that of Magri; the gradients of the conserved quantities are generators of the symmetries of an equation - hence so are the multipliers.

Fokas and Fuchssteiner (1981) also provide an operator-based study of symmetries and conservation laws. Their conserved covariants are the same as Magri's potential operators and thus, if multipliers exist, the conserved covariants are equivalent to multipliers.

Gradients are also mentioned by Lax (1968). He shows that

$$QK = \frac{dX}{dx} \quad (2.38)$$

for a number of conservation laws for the KdV.

c) Finding Multipliers from known Conservation Laws

If the conservation laws of an equation are already known the multipliers can be found from the conservation laws. In the case of an evolution equation one differentiates the density and manipulates the result to produce the product $M(u)u_t$; $M(u)$ is then the required multiplier. For other equations one performs the appropriate differentiations on all parts of the conservation law. The manipulations are carried out in a similar way to those

for evolution equations but it may be harder to find the required multiplier.

Ordinary Differential Equations

Ordinary differential equations involve only one independent variable (in the cases below the variable is time) and the conserved densities are called constants of the motion.

The system with two degrees of freedom and Lagrangian

$$L = 1/2 k(q_{1t}^2 + q_{2t}^2) - (aq_1 + bq_2) \quad (2.39)$$

has the equations of motion

$$kq_{1tt} + a = 0 \quad (2.40a)$$

$$kq_{2tt} + b = 0. \quad (2.40b)$$

Because this system has two degrees of freedom it will have four independent constants of the motion. (A well-known theorem in classical mechanics states that a system with n degrees of freedom has $2n$ independent constants of the motion.)

Four independent constants of the motion with their multipliers are

<u>Constant of the motion</u>	<u>Multiplier</u>
$1/2k(q_{1t}^2 + q_{2t}^2) + aq_1 + bq_2$	$q_{1t} \times (2.40a) + q_{2t} \times (2.40b)$
$k(bq_{1t} - aq_{2t})$	$b \times (2.40a) - a \times (2.40b)$
$kq_2 + bt$	$1 \times (2.40b)$
$q_1 - a/2k t^2 - q_{1t}t$	$- t/k \times (2.40a).$

Two other constants of the motion (not independent of those above) are

$$kq_{1t} + at \quad 1 \times (2.40a)$$

$$q_2 - b/2k t^2 - q_{2t} \quad t/k \times (2.40b). \quad (2.42)$$

Lutzky (1978) gives five constants of the motion for the harmonic oscillator

$$q_{tt} + q = 0. \quad (2.43)$$

Only two of these constants of the motion are independent since the equation has only one degree of freedom. (This equation is a good example of some of the problems associated with Noether's Theorem. The equation of motion has an eight-parameter symmetry group [Wulfman and Wybourne (1976)] yet the action integral (with which Noether's Theorem deals) has only a five parameter symmetry group [Lutzky (1978)]. Hence by Noether's Theorem there are five constants of the motion - but only two of these are independent!)

The five constants of the motion with their multipliers are

<u>Constant of the motion</u>	<u>Multiplier</u>	
$1/2(q_t^2 - q^2) \sin 2t - qq_t \cos 2t$	$q_t \sin 2t - q \cos 2t$	
$1/2(q_t^2 - q^2) \cos 2t \pm qq_t \sin 2t$	$q_t \cos 2t + q \sin 2t$	
$- 2q_t \cos t - 2q \sin t$	$-2 \cos t$	(2.44)
$- 2q_t \sin t + 2q \cos t$	$-2 \sin t$	
$q_t^2 + q^2$	$2q_t$	

Equation (2.43) is equation (2.10) with $w^2(t) = 1$. The constant of the motion given for (2.10) is equivalent to the fifth one above if $r = C = 1$.

Evolution Equations

1. Korteweg-de Vries Equation [McGuinness (1978,1980a,b), Miura et al (1968)]

$$u_t + u_x u + u_{xxx} = 0. \quad (2.45)$$

The first three conservation laws (other than the equation itself) have densities

$$1/2 u^2, 1/3 u^3 - u_x^2, 1/4 u^4 - 3u u_x^2 + 9/5 u_{xx}^2. \quad (2.46)$$

These are produced by the multipliers

$$u, u^2 + 2u_{xx}, u^3 + 3u_x^2 + 6u_{xx}u + 18/5 u_{xxxx}. \quad (2.47)$$

McGuinness (1978) shows that the conservation laws of the Korteweg-de Vries equation can be produced by operating on the equation in the form

$$u_{xt} + u_{xx}u_x + u_{xxxx} = 0 \quad (2.48)$$

with the operator H^n , $n = 0, 1, 2, \dots$ where

$$H = d^2/dx^2 + 2/3 u_x + 1/3 u_{xx} \int_{-\infty}^x dx \quad (2.49)$$

and multiplying the result by u_t .

One would expect a link between the operators and the multipliers which are guaranteed to exist for every conservation law by the theorem of Martínez Alonso in section b). Let A be an arbitrary function. If one uses the McGuinness operator once and multiplies by u_t one gets

$$u_t(d^2A/dx^2 + 2/3 u_x A + 1/3 u_{xx} \int_{-\infty}^x A dx). \quad (2.50)$$

Since the multipliers contain only x -derivatives the t -derivatives are eliminated by using the equation. McGuinness assumes that $u(x, t)$ and its derivatives are zero on the definite boundary of the integral, chosen here to be $-\infty$.

Thus $u_t = - \int_{-\infty}^x u_{xx} u_x + u_{xxxx} dx$ and

$$\begin{aligned}
& (u_x^2/2 + u_{xxx})(d^2A/dx^2 + 2/3 u_x A + 1/3 u_{xx} \int_{-\infty}^x A dx) \\
& = A d^2/dx^2(u_x^2/2 + u_{xxx}) + d/dx(dA/dx(u_x^2 + u_{xxx})) \\
& \quad - d/dx(A d/dx(u_x^2 + u_{xxx})) \\
& \quad + 1/3 u_x^3 A + 2/3 u_{xxx} u_x A \\
& \quad + d/dx(u_x^3/18 \int_{-\infty}^x A dx) - (u_x^3/18)A \\
& \quad + d/dx(u_{xx}^2/6 \int_{-\infty}^x A dx) - (u_{xx}^2/6)A \\
& = d/dx B + A (u_{xxxxx} + 5/6 u_{xx}^2 + 5/3 u_{xxx} u_x + 5/18 u_x^3).
\end{aligned} \tag{2.51}$$

Since every term in B contains A it can be seen that $B = 0$ for solutions to the equation $A = 0$. Remembering that the form of the equation used here is obtained by substituting u_x for u it can be seen that the operator $u_t H$ is equivalent to the third multiplier whenever the assumption of McGuinness on the form of u holds.

Miura et al (1968) also found a conservation law involving x and t explicitly

$$\begin{aligned}
& d/dt(xu - 1/2 tu^2) \\
& + d/dx(1/2 xu^2 + xu_{xx} - u_x - 1/3 tu^3 - tu_{xx}u + 1/2 tu_x^2) \\
& = 0.
\end{aligned} \tag{2.52}$$

This is produced by the multiplier $x-ut$.

2. Modified Korteweg-de Vries Equation [McGuinness (1980a), Miura et al (1968)]

$$u_t + u^2 u_x + u_{xxx} = 0 \tag{2.53}$$

The first three densities are

$$1/2 u_x^2, 1/4 u^4 - 3/2 u_x^2, 1/6 u^6 - 5u^2 u_x^2 + 3 u_{xx}^2 \tag{2.54}$$

and are produced by the multipliers

$$u, u^3 + 3u_{xx}, u^5 + 10u_x^2 u + 10u_{xx} u^2 + 6u_{xxxx}. \quad (2.55)$$

3. Sine-Gordon Equation [Sanuki and Konno (1974)]

$$u_{xt} - \sin u = 0 \quad (2.56)$$

The first three densities are

$$u_x^2, u_{xx}^2 + 1/4 u_x^4, u_{xxx}^2 - 5/2 u_{xx}^2 u_x^2 + 1/8 u_x^6 \quad (2.57)$$

and are produced by the multipliers

$$u_x, 2u_{xxx} + u_x^3, 2u_{xxxxx} + 5u_x^2 u_{xxx} + 5u_{xx}^2 u_x + 3/4 u_x^5. \quad (2.58)$$

If the substitution $u \rightarrow \sqrt{3/2} u_x$ is made in the multipliers of the Modified Korteweg-de Vries equation one obtains the multipliers of the Sine-Gordon equation. This point is discussed further in chapter four.

McGuinness (1980a) uses the integro-differential operators $u_x S^n$, $n = 0, 1, 2, \dots$ where

$$S = d^2/dx^2 + u_x^2 + u_{xx} \int_{-\infty}^x u_x dx \quad (2.59)$$

to produce the conservation laws.

By multiplying S^n by u_t instead of u_x he shows that the energy densities of the equations

$$S^n(u_{xt} - \sin u) = 0 \quad (2.60)$$

are all zero. To relate this to the multiplier approach replace u_t by $\int_{-\infty}^x \sin u dx$ (from the equation) and operate on A , an arbitrary function, with $(\int_{-\infty}^x \sin u dx)S$. One gets

$$(\int_{-\infty}^x \sin u dx)(d^2 A/dx^2 + A u_x^2 + u_{xx} \int_{-\infty}^x u_x A dx)$$

$$\begin{aligned}
&= d/dx(dA/dx \int_{-\infty}^x \sin u \, dx) - d/dx(A \sin u) + Au_x \cos u \\
&\quad + Au_x^2 \int_{-\infty}^x \sin u \, dx \\
&\quad + d/dx(u_x (\int_{-\infty}^x \sin u \, dx) \int_{-\infty}^x u_x A \, dx) \quad (2.61) \\
&\quad + d/dx(\cos u \int_{-\infty}^x u_x A \, dx) - Au_x \cos u - Au_x^2 \int_{-\infty}^x \sin u \, dx \\
&= dB/dx \\
&= 0 \text{ when } A = 0.
\end{aligned}$$

Thus the operator $u_t S$ is equivalent to the multiplier zero and therefore all $u_t S^n$, $n = 0, 1, 2, \dots$ are equivalent to zero.

The Sine-Gordon equation is not of the form for the theorem of Martínez Alonso to apply. If, however, one makes the substitution $u_x \rightarrow v$ and one assumes that u and its x -derivatives are zero at $-\infty$, one gets

$$v_t - \sin \int_{-\infty}^x v \, dx = 0. \quad (2.62)$$

The multipliers and conservation laws could be similarly transformed. For this form of the equation the theorem of Martínez Alonso does apply and every conservation law has a multiplier.

However the assumption about the boundary conditions means that the existence of a multiplier for $u_{xt} - \sin u = 0$ for every conservation law can only be proved for certain solutions - those which, along with their derivatives, are zero on the boundaries. Nevertheless it seems likely that multipliers exist for all conservation laws for all solutions of the Sine-Gordon equation. The operators of McGuinness also are only proved to produce conservation laws for solutions which are zero on the boundary

yet the conservation laws are clearly seen to be valid for all solutions and the multipliers given above clearly exist for all solutions. Thus if a multiplier exists one would expect it to be valid for all solutions and since multipliers and operators are closely related one would also expect a multiplier for every operator - thus one can reasonably conjecture that multipliers exist for all solutions and for every conservation law of the Sine-Gordon equation.

4. Non-linear Water Waves

A system of equations with an infinite number of conservation laws is the pair of equations for non-linear water waves [McGuinness (1980a)]

$$h_t + uh_x + hu_x = 0 \quad (2.63a)$$

$$u_t + uu_x + h_x = 0. \quad (2.63b)$$

This is a special case of the long water waves studied by Benney (1973).

The first three densities are

$$uh, 1/2 u^2h + 1/2 h^2, 1/3 u^3h + uh^2. \quad (2.64)$$

Here the multipliers take the form of those in (2.41). Each equation is multiplied by a different multiplier and the products are added. The multipliers for the three densities given above are

$$\begin{aligned} & u \times (2.63a) + h \times (2.63b) \\ & (h + 1/2 u^2) \times (2.63a) + uh \times (2.63b) \end{aligned} \quad (2.64)$$

$$(2uh + 1/3 u^3) \times (2.63a) + (u^2h + h^2) \times (2.63b).$$

Integro-differential operators can be used to produce the conservation laws of the above four equations [McGuinness (1980a,b)]. These operators are proven to be valid only for a restricted set of solutions. On the other hand, polynomial multipliers have been found which are valid for all solutions. For the restricted set of solutions the multipliers and operators can be shown to be equivalent by using the procedure used for the Korteweg-de Vries equation.

5. Non-Linear Schrödinger Equation [Zakharov and Shabat (1972)]

$$iu_t + u_{xx} + u^2 u^* = 0 \quad (2.65)$$

Multipliers can be found for this equation if we consider the system of the equation and its complex conjugate:

$$\begin{aligned} E : iu_t + u_{xx} + u^2 u^* &= 0 \\ E^* : -iu_t^* + u_{xx}^* + u^{*2} u &= 0. \end{aligned} \quad (2.66)$$

The first three densities are

$$uu^*, u^* u_x - uu_x^*, u_x u_x^* - 1/2 (uu^*)^2. \quad (2.67)$$

These are produced by the multipliers

$$\begin{aligned} &iu^* E - iu E^* \\ &iu_x^* E + iu_x E^* \\ &-(uu^{*2} + u_{xx}^*) E + (u^* u^2 + u_{xx}) E^*. \end{aligned} \quad (2.68)$$

6. Kadomtsev-Petviashvili Equation

$$u_{xt} + u_{xx}u_x + u_{xxxx} + u_{yy} = 0 \quad (2.69)$$

No infinite sets of polynomial conservation laws have been found for this equation. It does however have at least two infinite sets of conservation laws; one containing integrals [Zakharov and Shulman (1980)] and one involving x , y , and t explicitly [Infeld and Frycz (1983)].

The polynomial densities

$$u_x^2, u_x u_y, u_x^3 - 3u_{xx}^2 + 3u_y^2 \quad (2.70)$$

can be produced by the multipliers u_x , u_y and u_t respectively. It is possible that these are the only polynomial multipliers.

The first three densities containing x , y and t explicitly are

$$\begin{aligned} &tu_x^2 - 2xu_x \\ &t^2/2 u_x^2 - 2xtu_x + y^2u_x \\ &tu_x u_y - 1/2 yu_x^2 \end{aligned} \quad (2.71)$$

and are produced by the multipliers

$$\begin{aligned} &2tu_x - 2x \\ &t^2u_x - 2xt + y^2 \\ &2tu_y - yu_x. \end{aligned} \quad (2.72)$$

Examples of multipliers are given below for two equations which are not differential equations - one is an integro-differential equation and the other is a differential-difference equation.

8. Benjamin-Ono Equation [Bock and Kruskal (1979)]

$$u_t + 2uu_x + H(u_{xx}) = 0 \quad (2.73)$$

where H is the Hilbert transform

$$H(q(x)) = \int \frac{q(z)}{z-x} dz, \quad (\text{the integral is the usual principal value}). \quad (2.74)$$

The first three densities are

$$u^2, u^3 + 3/2 uH(u_x), u^4 + 3u^2H(u_x) + 2u_x^2 \quad (2.75)$$

and are produced by the multipliers

$$2u, 3u^2 + 3/2 H(u_x) - 3/2 u_x H, 4u^3 + 6uH(u_x) - 6uu_x H - 4u_{xx}. \quad (2.76)$$

9. Toda Differential-Difference Equation [Ablowitz and Ladik (1976)]

$$u_{n,tt} - e^{-(u_n - u_{n-1})} + e^{-(u_{n+1} - u_n)} = 0 \quad (2.77)$$

where $n = -\infty, \dots, 0, \dots, \infty$.

The three densities are

$$\sum_{n=-\infty}^{\infty} u_{n,t}$$

$$\sum_{n=-\infty}^{\infty} (1/2 u_{n,t}^2 + e^{-(u_n - u_{n-1})}) \quad (2.78)$$

$$\sum_{n=-\infty}^{\infty} 1/3 u_{n,t}^3 + u_{n,t} (e^{-(u_n - u_{n-1})} + e^{-(u_{n+1} - u_n)}).$$

The first is derived from the equation by summing over all n . The exponential terms cancel. The second is obtained by multiplication by $u_{n,t}$ and summation and the third by multiplication by

$$u_{n,t} - e^{-(u_n - u_{n-1})} + e^{-(u_{n+1} - u_n)} \quad (2.79)$$

and summation.

d) My Previous Work

The method of multipliers is a useful method of producing conservation laws and multipliers exist for an extremely wide range of (perhaps for all) partial differential equations.

In an earlier project [Suttie (1981)] I attempted to develop a method of finding multipliers for polynomial equations in one variable depending on x and t . Although a method was developed, it was too cumbersome to be widely used. However, it showed that such a method was feasible and it pointed the way to the work presented in this thesis.

The method that was developed will now be outlined.

A conservation law $d_m J^m = 0$ will, in general, have terms which differ in the number of factors of u and in the numbers of derivatives. For example consider the conservation law (2.7). All terms have the same number of u 's but the first term has three t -derivatives (counting the one outside the brackets) while the others have two t -derivatives and one x -derivative.

Define the sub-rank of a term to be the set of numbers (number of factors of u , number of x -derivatives, number of t -derivatives).

For example, the sub-rank of $u_{xx}u_{xt}u_t$ is $(3,3,2)$.

(In Suttie (1981) the term total cardinality was used instead of sub-rank, but sub-rank will be used here to be consistent with the nomenclature of chapter four.)

Given a polynomial, K , one can divide it into parts each of which contain all terms with a particular sub-rank. Thus

$$K = \sum_i K_i \quad (2.80)$$

where K_i contains all terms with the sub-rank (a_i, b_i, c_i) . If K is to equal a divergence each K_i must separately equal a divergence since terms of different sub-ranks cannot combine to form divergences. For example, for the polynomial

$$u_{xxx}u_t + u_{xx}u_{xt} + u_{ttx}u_x + u_{tt}u_{xx} \quad (2.81)$$

the terms $u_{xxx}u_t$ and $u_{xx}u_{xt}$ have sub-rank $(2,3,1)$ and $u_{ttx}u_x$ and $u_{tt}u_{xx}$ have sub-rank $(2,2,2)$. This equals a divergence since

$$u_{xxx}u_t + u_{xx}u_{xt} = (u_{xx}u_t)_x \quad (2.82)$$

$$\text{and } u_{ttx}u_x + u_{tt}u_{xx} = (u_{tt}u_x)_x. \quad (2.83)$$

Thus given a multiplier for an equation the product of equation and multiplier can be divided into parts of different sub-rank and each part must equal a divergence if a conservation law is to be produced.

If one is searching for multipliers it is obviously absurd to try to check possible multipliers (of which there is an infinite number) one at a time. The task can be simplified dramatically if, instead of individual multipliers, one considers multipliers with terms which have undetermined coefficients and which have a particular set of sub-ranks. For example, a multiplier may have

terms with the sub-ranks (a_1, b_1, c_1) , (a_2, b_2, c_2) and (a_3, b_3, c_3) . Conditions on the undetermined coefficients of the terms can then be found in order for conservation laws to be formed. If, say, (a_1, b_1, c_1) is chosen first, then it is possible to choose other sub-ranks in a systematic way such that all multipliers which have terms with sub-rank (a_1, b_1, c_1) can be found. The method used in Suttie (1981), was very restrictive - a more general one is given in chapter four.

For each sub-rank there is a number of terms which have that sub-rank. For example, for the sub-rank $(3, 2, 1)$ one has the following terms

$$u_{xxt}u^2, u_{xx}u_tu, u_x^2u_t, u_{xt}u_xu. \quad (2.84)$$

Any polynomial whose terms all have the same sub-rank can be represented by a vector

$$\begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \quad (2.85)$$

where the a_i are the coefficients (which may be zero) of the various terms. (For uniqueness one must choose a particular ordering of the terms.) If a polynomial, P , has terms with sub-rank (a, b, c) and if it can be written as a divergence

$$P = \frac{dT}{dt} + \frac{dX}{dx} \quad (2.86)$$

then the elements of T have sub-rank $(a, b, c-1)$ and those of X

sub-rank $(a, b-1, c)$.

Let P be represented by the vector \underline{p} , T by \underline{t} and X by \underline{x} and let \underline{b} be the vector

$$\begin{bmatrix} \underline{x} \\ \underline{t} \end{bmatrix} \quad (2.87)$$

It is shown in Suttie (1981) that these vectors can be related by an equation

$$\underline{p} = \underline{D} \underline{b} \quad (2.88)$$

where \underline{D} is a matrix with integer elements called the differentiation matrix.

For example, let \underline{p} have sub-rank $(3, 3, 1)$. \underline{x} has sub-rank $(3, 2, 1)$ and \underline{t} has sub-rank $(3, 3, 0)$. \underline{p} will be represented by the vector

$$\begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_6 \end{bmatrix} \quad (2.89)$$

$$\text{where } \underline{p} = p_1 u_{xxx} t^2 + p_2 u_{xxx} u_t u + p_3 u_{xxt} u_x u + p_4 u_{xx} u_{xt} u + p_5 u_{xx} u_x u_t + p_6 u_{xt} u_x^2. \quad (2.90)$$

Similarly \underline{x} is represented by

$$\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_4 \end{bmatrix} \quad (2.91)$$

$$\text{where } \underline{x} = x_1 u_{xxt} u^2 + x_2 u_{xx} u_t u + x_3 u_{xt} u_x u + x_4 u_x^2 u_t \quad (2.92)$$

and \underline{t} is represented by

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \quad (2.93)$$

$$\text{where } \underline{t} = t_1 u_{xxx} u^2 + t_2 u_{xx} u_x u + t_3 u_x^3. \quad (2.94)$$

One now gets

$$\underline{b} = \begin{bmatrix} x_1 \\ \cdot \\ x_4 \\ t_1 \\ \cdot \\ t_3 \end{bmatrix} \quad (2.95)$$

and

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} \quad (2.96)$$

To see how the differentiation matrix is derived, consider the coefficient of $u_{xxx} u^2$, that is, p_1 . The only terms in \underline{b} which produce $u_{xxx} u^2$ when differentiated are $x_1 u_{xxt} u^2$ and $t_1 u_{xxx} u^2$. Differentiating these terms one gets

$$\begin{aligned} & d/dx(x_1 u_{xxt} u^2) + d/dt(t_1 u_{xxx} u^2) \\ &= (x_1 + t_1) u_{xxx} u^2 + \text{other terms} \quad (2.97) \\ &= p_1 u_{xxx} u^2 + \text{other terms.} \end{aligned}$$

$$\begin{aligned}
 \text{Thus } p_1 &= x_1 + t_1 \\
 &= b_1 + b_5.
 \end{aligned}
 \tag{2.98}$$

This is represented by the first row of the differentiation matrix.

The elements of \underline{b} are arbitrary since every polynomial can be differentiated while those of \underline{p} are not since not every polynomial is equal to a divergence.

Thus one must find which values of p_1, \dots, p_6 correspond to polynomials which are equal to divergences. This is done by finding a matrix \underline{M} such that

$$\underline{M} \underline{D} = \begin{pmatrix} \underline{I} & \underline{K} \\ \underline{0} & \underline{0} \end{pmatrix}
 \tag{2.99}$$

where \underline{I} is the identity matrix, $\underline{0}$ is the zero matrix and \underline{K} is arbitrary. This is equivalent to performing row operations on \underline{D} in order to find its row-echelon form. This can always be done and thus \underline{M} will always exist [Ben-Israel and Greville (1974)]. If there were no zero rows in $\underline{M} \underline{D}$ every \underline{p} would be a divergence - this occurs for linear terms with one or more derivatives and for quadratic terms with an odd number of derivatives.

If there are n zero rows then there are n conditions on the p_1 since

$$\underline{M}' \underline{p} = (\underline{0}) \underline{b} = 0
 \tag{2.100}$$

where \underline{M}' is formed from the bottom n rows of \underline{M} . These n conditions are also sufficient for \underline{p} to be a divergence [Suttie(1981)]. A solution to these n conditions is guaranteed by

the fact that \underline{x} and \underline{t} can always be differentiated to produce a \underline{p} .

In order to use this method to find multipliers for an equation choose a set of sub-ranks and calculate the possible terms having those sub-ranks. Multiply the equation by these terms and split the product into its different sub-ranks. The coefficients in the product will depend on the coefficients of the equation and the (undetermined) coefficients of the multiplier. For example

$$\begin{aligned}
 & (au_{xx}u_x + bu_{xxx}u + cu_x^2 + du_{xxu}) (u_{xx} + 2u_x) \\
 & = (au_{xx}^2u_x + bu_{xxx}u_{xx}u) \\
 & \quad + ((2a+c)u_{xx}u_x^2 + 2bu_{xxx}u_xu + du_{xx}^2u) \\
 & \quad + (2cu_x^3 + 2du_{xx}u_xu).
 \end{aligned} \tag{1.101}$$

One can calculate the conditions on the coefficients of the multiplier a, b, c and d using the method described above. One finds that there are no conservation laws for the subranks chosen.

The overall condition on the multiplier for a conservation law to exist will be that the conditions for each sub-rank all hold simultaneously. Although for each individual sub-rank a solution is guaranteed this is not necessarily the case for the multiplier as a whole.

A computer program was developed which used this algorithm. The program was in FORTRAN for use on a Burroughs B6700 machine. Unfortunately the size of the differentiation matrix quickly

became very large as the order of the multiplier was increased and the available storage space of the computer was exceeded before significant results could be obtained. A different but more workable algorithm has been developed during the course of this project and is described in chapter five.

CHAPTER THREE

Finding conservation laws using the variational derivative.

It is well-known that if a polynomial $K(u, u_x, u_t, \dots)$ is equivalent to a divergence, that is if

$$K(u) = \frac{dX}{dx} + \frac{dT}{dt} \quad (3.1)$$

then

$$\frac{\delta}{\delta u} K(u) = 0 \quad (3.2)$$

where

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - \frac{d}{dx} \frac{\partial}{\partial u_x} - \frac{d}{dt} \frac{\partial}{\partial u_t} + \frac{d^2}{dx^2} \frac{\partial}{\partial u_{xx}} \\ & + \frac{d^2}{dx dt} \frac{\partial}{\partial u_{xt}} + \frac{d^2}{dt^2} \frac{\partial}{\partial u_{tt}} - \dots \end{aligned} \quad (3.3)$$

If one is trying to find the multipliers for an equation then one often needs to know if the product of two polynomials is equivalent to a divergence. Thus if one has an equation

$$G(u) = 0 \quad (3.4)$$

and one wishes to multiply it by $F(u)$ to produce a conservation law one needs to know if FG is a divergence. In other words, does the following hold,

$$\frac{\delta}{\delta u} (FG) = 0 ? \quad (3.5)$$

Galindo (1979b) provides a formula for $\delta/\delta u(FG)$:

$$\frac{\delta}{\delta u} (FG) = \sum_{i=0}^{\infty} (-1)^i [(D^i F) \frac{\delta G}{\delta u_i} + (D^i G) \frac{\delta F}{\delta u_i}] \quad (3.6)$$

where

$$D^i = (D^x)^{i_x} (D^t)^{i_t}$$

$$i = i_x + i_t; i, i_x, i_t \text{ integer}$$

$$D^x = \sum_{j=0}^{\infty} u_{jx} \frac{\partial}{\partial u_j}, \quad D^t = \sum_{j=0}^{\infty} u_{jt} \frac{\partial}{\partial u_j}$$

$$u_i = \frac{\partial^{i_x} u}{\partial x^{i_x} \partial t^{i_t}} \quad (3.7)$$

$$\text{and} \quad \frac{\delta}{\delta u_i} = \sum_{j=i}^{\infty} \binom{j}{i} \frac{d^{j-i}}{dx^{j-i}} \frac{\partial}{\partial u_j}$$

$$\binom{j}{i} = \frac{j!}{i_t!(j-i_t)!} \times \frac{j!}{i_x!(j-i_x)!}.$$

In section a) this formula will be used to obtain a number of previously found results. Using one of the results from a) a theorem for the equation $u_t + u_n + R = 0$ will be proved in b).

a) Previously found results

(1) Linear Evolution Equations

The above formula can be used to derive the polynomial conservation laws of linear evolution equations

$$G = \frac{\partial u}{\partial t} + \sum_i a_i \frac{\partial^{i_x} u}{\partial x^{i_x}} = 0, \quad a_i \text{ constant.} \quad (3.8)$$

Any multipliers for such equations will be functions of u and its x -derivatives since any t -derivatives can be eliminated

by using the equation.

Multiply G by a multiplier $F(u, u_x, \dots)$, an arbitrary polynomial in u and its x -derivatives - since polynomial conservation laws are being studied, F will be polynomial. The product FG will equal a divergence if and only if $\delta/\delta u (FG) = 0$.

Using the formula for $\delta/\delta u (FG)$ given above one gets

$$\begin{aligned}
 -\frac{dF}{dt} + \sum_{n=0}^{\infty} (-1)^n a_n \frac{d^n F}{dx^n} + \sum_{i=0}^{\infty} (-1)^i u_{ti} \frac{\delta F}{\delta u_i} \\
 + \sum_{n=0}^{\infty} \sum_{i=0}^n (-1)^i a_n (u_{n+i}) \frac{\delta F}{\delta u_i} = 0
 \end{aligned} \tag{3.9}$$

The only terms containing t -derivatives are

$$-\frac{dF}{dt} \quad \text{and} \quad \sum_{i=0}^{\infty} (-1)^i u_{ti} \frac{\delta F}{\delta u_i} \tag{3.10}$$

Thus

$$-\frac{dF}{dt} + \sum_{i=0}^{\infty} (-1)^i u_{ti} \frac{\delta F}{\delta u_i} = 0. \tag{3.11}$$

This is a condition on F for the existence of a conservation law.

In general F may contain terms with differing numbers of factors of u . Since a linear equation is being considered terms in F differing in the number of factors of u cannot combine to form divergences and thus they must form separate multipliers. One may therefore assume, without loss of generality, that all terms in F have the same number of factors of u .

Terms in the product FG will differ in the number of x -derivatives. Each sum which contains all the terms with a particular number of x -derivatives must equal a divergence.

Let $F = F_1 + \dots + F_m$ where each F_j contains terms with the same number of x -derivatives, F_m having the greatest number. If

$\partial^n u / \partial x^n$ is the equation term with the highest number of x -derivatives then the product $F_m (\partial^n u / \partial x^n)$ must equal a divergence. From (3.6) one gets

$$(-1)^n \frac{d^n F_m}{dx^n} + \sum_{i=0}^{\infty} (-1)^i u_{n+i} \frac{\delta F_m}{\delta u_i} = 0 \quad (3.12)$$

and from (3.11) one gets

$$-\frac{dF_j}{dt} + \sum_{i=0}^{\infty} (-1)^i u_{ti} \frac{\delta F_j}{\delta u_i} = 0 \quad (3.13)$$

for $j = 1, \dots, m$.

If

$$\frac{dF_m}{dt} = \sum_{i=0}^{\infty} (-1)^i u_{ti} \frac{\delta F_m}{\delta u_i} \quad (3.14)$$

then

$$(-1)^n \frac{d^n F_m}{dx^n} \neq - \sum_{i=0}^{\infty} (-1)^i u_{n+i} \frac{\delta F_m}{\delta u_i} \quad (3.15)$$

unless $n = 1$ or $\delta F_m / \delta u_i = \text{constant}$.

This is because replacing d/dt in each term of dF_m/dt by d^n/dx^n cannot produce $\pm d^n F_m/dx^n$ unless $n = 1$.

Thus if $n \neq 1$ then F_m must be linear in u and $F = \sum_m a_m u_m$, a_m constant.

If $n = 1$ then the equation is $u_t + u_x = 0$ and it is well known that for this equation every function of u and its derivatives is a conserved density.

If F is linear then

$$\frac{\delta F}{\delta u_i} = \begin{cases} a_i & \text{if } F \text{ contains } u_i \\ 0 & \text{if } F \text{ does not contain } u_i. \end{cases} \quad (3.16)$$

Since each F_j contains a different number of x -derivatives each $F_j u_t$ must equal a divergence.

If $F_j = u_k$ then

$$-\frac{\partial u_k}{\partial t} + (-1)^k \frac{\partial^k u_t}{\partial x^k} = 0 \quad (3.17)$$

\Rightarrow k must be even.

Thus all terms in F_j must contain an even number of x -derivatives (in this case define F_j to be 'even').

If all the F_j are even then even terms in the equation cannot combine with odd terms to form divergences. Thus the even terms form divergences separately from the odd terms. This means that the even term in the equation with the highest number of x -derivatives must equal a divergence when multiplied by F_n . This is impossible since $u_a u_b$ where a, b are even cannot equal a divergence. On the other hand, every odd term multiplied by an even term equals a divergence.

Thus the only linear evolution equations which have multipliers for polynomial conservation laws are $u_t + \sum_m a_m u_m$ where m is odd. For $m > 1$, the only multipliers are u_i , $i = 0, 2, 4, \dots$

The conservation laws obtained are identical to those found by Abellanas and Galindo (1978). The method used here is much more direct than that used by these authors as it does not

require a special formalism.

The results of this section generalise to the case of more than two independent variables. In this case a term is even (odd) if the total number of derivatives is even (odd). A single term is equal to a divergence if and only if it is odd. This can be proved as follows. Consider a series of manipulations on a term

with two factors $u_{x_1 x_2 \dots x_m} u_{x_{m+1} \dots x_N}$. Manipulations of the form

$$u_{x_1 x_2 \dots x_m} u_{x_{m+1} \dots x_N} = (u_{x_2 \dots x_m} u_{x_{m+1} \dots x_N})_{x_1} - u_{x_2 \dots x_m} u_{x_1 x_{m+1} \dots x_N} \quad (3.18)$$

will result in equations of the form

$$u_{x_1 \dots x_m} u_{x_{m+1} \dots x_N} = \text{divergence} + (-1)^n u_{y_1 \dots y_N} \quad (3.19)$$

where $y_1 \dots y_N$ is some arrangement of the variables $x_1 \dots x_N$ and n is the number of manipulations. If, after n such manipulations,

$u_{y_1 \dots y_N}$ is the same as $u_{x_1 \dots x_m} u_{x_{m+1} \dots x_N}$ and if n is odd, then clearly the original general term is a divergence. If n is even however, the non-derivative terms on the two sides of (3.19) cancel leaving the identity $0 = 0$ and providing no information on the original term.

Each such manipulation moves a derivative from one of the two factors to the other. If all the derivatives are with respect to different variables then, for the non-derivative term on the right-hand side of (3.19) to be the same as that on the left-hand side, every derivative on the first factor must be moved to the second factor and vice-versa. To move all the derivatives from their original factors to the other will require $n = N$ manipulations. Thus, if N is odd, the original term is equivalent to a divergence.

If the variables x_i are not all different, then fewer than N steps may be necessary to produce the same non-derivative term on both sides of equation (3.19). For example,

$$u_{xy} u_y = (u_y u_y)_x - u_y u_{yx} \quad (3.20)$$

so that these three derivatives need just one manipulation to give

$$u_{xy}u_y = \frac{1}{2}(u_y^2)_x. \quad (3.21)$$

When the same derivative occurs in both factors a manipulation is saved in each direction. Thus the number of steps saved will always be even and if N is odd the number of manipulations needed will remain odd.

If N is even however, any attempt to express a general term $u_{x_1 \dots x_m} u_{x_{m+1} \dots x_N}$ as $\sum_{i=1}^N \frac{\partial}{\partial x_i} \sum_k a_{ik} u_{y_1 \dots y_j} u_{y_{j+1} \dots y_{N-1}}$, where the $y_1 \dots y_{N-1}$ is the k th combination of the $x_1 \dots x_N$ excluding x_i and the a_{ik} are numerical constants, leads immediately to a set of equations for the a_{ik} which are not consistent.

Thus $u_{x_1 \dots x_m} u_{x_{m+1} \dots x_N}$ is equivalent to a divergence if and only if the total number of derivatives, N, is odd.

For non-evolution equations proofs of the above simple type are not possible because the multiplier may depend on t-derivatives and the problem cannot then be split into two separate parts as was done here.

(ii) Non-Linear Equations

The methods applied to linear equations in the previous section can be applied to some non-linear equations. For example, Whitham (1974) discusses the equation

$$\frac{\partial r}{\partial t} + c(r) \frac{\partial r}{\partial x} = 0 \quad (3.22)$$

where c is any integrable function of r.

He shows that there are an infinite number of conservation laws

$$\frac{d}{dt} f + \frac{d}{dx} g = 0 \quad (3.23)$$

where f, g are functions of r and

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial r} c(r). \quad (3.24)$$

This equation will now be considered using the formula for $\delta/\delta u$ (FG).

Let F , a function of r and its x -derivatives, be a multiplier for the above equation.

$$\text{Then } -\frac{dF}{dt} + \sum_{i=0}^{\infty} (-1)^i r_{ti} \frac{\delta F}{\delta r_i} = 0 \quad (3.25)$$

$$\text{and } -c \frac{dF}{dx} + \sum_{i=0}^{\infty} (-1)^i (cr_x)_i \frac{\delta F}{\delta r_i} = 0. \quad (3.26)$$

Since F contains no t -derivatives

$$\text{if } \frac{dF}{dt} = \sum_{i=0}^{\infty} (-1)^i r_{ti} \frac{\delta F}{\delta r_i} \quad (3.27)$$

$$\text{then } c \frac{dF}{dx} = \sum_{i=0}^{\infty} (-1)^i cr_{xi} \frac{\delta F}{\delta r_i}$$

$$\Rightarrow \sum_{i=0}^{\infty} (-1)^i (cr_x)_i \frac{\delta F}{\delta r_i} = \sum_{i=0}^{\infty} (-1)^i cr_{xi} \frac{\delta F}{\delta r_i} \quad (3.28)$$

$$\Rightarrow \frac{\delta F}{\delta r_i} = 0 \text{ for } i \neq 0$$

(unless $c = c_0 = \text{constant}$ in which case one gets $r_t + c_0 r_x = 0$)

$$\Rightarrow F \text{ is a polynomial in } r \text{ only.} \quad (3.29)$$

Each term in F can be treated as a separate multiplier and one gets

$$F = r^n. \quad (3.30)$$

Thus one gets an infinite set of multipliers r^n , $n = 0, 1, 2, \dots$

and the conservation laws are

$$\left(\frac{1}{n+1} r^{n+1} \right)_t + \left(\int r^n c(r) dr \right)_x = 0. \quad (3.31)$$

Comparing this with Whitham's result one has

$$f = \frac{r^{n+1}}{n+1}, \quad g = \int r^n c(r) dr \quad (3.32)$$

and indeed $\frac{\partial g}{\partial r} = \frac{\partial f}{\partial r} c(r)$ as required.

Guil Guerrero and Martínez Alonso (1980) consider the equation

$$u_t - (u^p)_{xx} = 0, \quad p \neq 0. \quad (3.33)$$

They show that the conserved densities are of the forms

$$\begin{aligned} p \neq 1, (ax+b)u \quad a, b \text{ real}, \\ p = 1, s(x,t)u \quad s \text{ a solution of } s_t + s_{xx} = 0. \end{aligned} \quad (3.34)$$

Consider multipliers for this equation which are explicit functions of x and t and which do not depend on u . If E is the multiplier then

$$Eu_t - Ep(p-1)u^{p-2}u_x^2 - Epu^{p-1}u_{xx} = 0. \quad (3.35)$$

If $p > 1$ then Eu_t will form a divergence separately from the rest of the equation since the number of factors of u is different.

$$\text{Thus } \frac{\delta}{\delta u}(Ep(p-1)u^{p-2}u_x^2 + Epu^{p-1}u_{xx}) = 0. \quad (3.36)$$

This reduces to

$$\begin{aligned} E_{xx}pu^{p-1} &= 0 \\ \Rightarrow E_{xx} &= 0 \\ \Rightarrow E &= A(t)x + B(t). \end{aligned} \quad (3.37)$$

But

$$\begin{aligned} \frac{\delta}{\delta u}(Eu_t) &= 0 \\ \Rightarrow \frac{d}{dt} \frac{\partial}{\partial u_t}(Eu_t) &= 0 \\ \Rightarrow \frac{dE}{dt} &= 0 \\ \Rightarrow A(t) \text{ and } B(t) &\text{ are constants.} \end{aligned} \quad (3.38)$$

Thus

$$E = ax + b \quad (3.39)$$

where a, b are constants and the conserved densities are of the form

$$(ax + b)u. \quad (3.40)$$

If $p = 1$ the equation is the Diffusion Equation and one gets

$$\begin{aligned} \frac{\delta}{\delta u} (Eu_t - Eu_{xx}) &= 0 \\ \Rightarrow -E_t - E_{xx} &= 0 \\ \Rightarrow E \text{ is any solution } s \text{ of } s_t + s_{xx} &= 0. \end{aligned} \quad (3.41)$$

Thus even though only functions of x and t were considered as multipliers the (completely general) results of Guil Guerrero and Martínez Alonso (1980) have been reproduced. This suggests that the only multipliers for the equation (3.33) are those found above. If there are others they must be equivalent to those found since they would produce the same conserved densities.

b) Evolution Equations of the form

$$\underline{u_t + u_n + R = 0}$$

R is any polynomial in u and its x -derivatives. It will be assumed (in complete generality) that the non-zero constant $a = 1$.

(i) $n = 1$

In this case one gets

$$u_t + u_x + R = 0. \quad (3.42)$$

Since this is an evolution equation, a multiplier, F , will

not contain t -derivatives. Thus (3.11) holds and, substituting x for t , one gets

$$\frac{dF}{dx} = \sum_i (-1)^i u_{xi} \frac{\delta F}{\delta u_i}. \quad (3.43)$$

Hence if F is a multiplier for u_t it will also be a multiplier for u_x and the multipliers for $u_t + R = 0$ will also be multipliers for

$$u_t + u_x + R = 0. \quad (3.44)$$

If one applies the transformation $u \rightarrow u + 1$ to the Korteweg-de Vries equation, one gets

$$\begin{aligned} & u_t + u_x u + u_{xxx} = 0 \\ \rightarrow & (u+1)_t + (u+1)_x (u+1) + (u+1)_{xxx} = 0 \\ \rightarrow & u_t + u_x + u_x u + u_{xxx} = 0. \end{aligned} \quad (3.45)$$

This is the form of the Korteweg-de Vries equation first studied by Korteweg and de Vries (to within the transformation $x \rightarrow ax$, $t \rightarrow bt$, $u \rightarrow cu$ where a, b and c are constants related to the physical constants of the equation).

This new equation has an infinite number of conservation laws (the transformed versions of those of the Korteweg-de Vries equation) each of which has a multiplier by the theorem of Martínez Alonso (see chapter two, section b). This set of multipliers must be related in some way (other than through the transformation used above) to those of the Korteweg-de Vries since it has just been shown that the multipliers for the Korteweg-de Vries equation are also multipliers for the new equation. If one transforms the three multipliers given in

chapter two one gets

$$\begin{aligned}
 u &\rightarrow u + 1 \\
 u^2 + 2u_{xx} &\rightarrow u^2 + 2u_{xx} + (2u + 1) \\
 u^3 + 3u_x^2 + 6u_{xx}u + 18/5 u_{xxxx} & \\
 \rightarrow u^3 + 3u_x^2 + 6u_{xx}u + 18/5 u_{xxxx} + 3(u^2 + 2u_{xx}) + (3u + 1). &
 \end{aligned} \tag{3.46}$$

Thus the n th transformed multiplier consists of a sum of the untransformed multipliers (that is, the multipliers of the untransformed Korteweg-de Vries equation) up to and including the n th. The densities and fluxes for the transformed version will also be sums of those of the untransformed Korteweg-de Vries equation.

The equation

$$u_t + u_{xxx} - 6u^2u_x + 6ku_x = 0 \tag{3.47}$$

is studied by Fung and Au (1982,84). In this case $u_t + R = 0$ is the Modified Korteweg-de Vries equation. This has an infinite number of polynomial conservation laws and multipliers but there is no simple transformation between the two equations as there was for the Korteweg-de Vries equation. (However, the Miura transformation, $u = -(v_x + v^2)$ where u is a solution of the KdV equation and v is a solution of the MKdV equation, may provide a link between the results for equation (3.47) and those for the transformed version of the KdV equation.)

Thus the equation (3.47) has an infinite number of conservation laws produced by the multipliers of the Modified Korteweg-de Vries equation. The conserved densities produced by

the multipliers will be the same as those of the Modified Korteweg-de Vries equation [Miura et al (1968)] but the fluxes will be different - they will contain additional terms produced by the product of the multiplier and the term u_x . For example, for the third multiplier given in chapter two for the Modified Korteweg-de Vries equation, one gets the additional terms

$$\begin{aligned} & (u^5 + 10u_x^2 u + 10u_{xx}u^2 + u_{xxxx})u_x \\ & = 1/6 (u^6)_x + 5(u_x^2 u^2)_x + (u_{xxx}u_x)_x - 1/2 (u_{xx}^2)_x. \end{aligned} \quad (3.48)$$

If R is linear there will be no multipliers other than those already found. If R is non-linear (as in the two examples above) there could exist other multipliers because the term u_x may combine with the non-linear terms in R to form divergences.

(ii) $n \neq 1$

Let $F = F_1 + \dots + F_j$ be a multiplier for a polynomial conservation law. F will be a function only of u and its x -derivatives.

Let $R_1 + \dots + R_m$ be the linear terms in R (the analysis still holds if R has no linear terms).

If $F_1 + \dots + F_k$ are the terms in F which contain the lowest number of factors of u then

$$(R_1 + \dots + R_m + u_n) \times (F_1 + \dots + F_k) \quad (3.49)$$

must equal a divergence. $u_t(F_1 + \dots + F_k)$ must also equal a divergence.

From section a) it can be seen that if any of the terms R_1 ,

..., R_m , u_n is even there will be no multipliers.

Thus the equation

$$u_t + au_n + R = 0 \quad (3.50)$$

where n is even and a is a non-zero constant, has at most one polynomial conservation law, the equation itself.

Two well-known equations of this form are Burger's Equation

$$u_t + u_x u - u_{xx} = 0 \quad (3.51)$$

and the Diffusion Equation

$$u_t - u_{xx} = 0. \quad (3.52)$$

Equations of this type can have non-polynomial conservation laws. For example, in section a) the Diffusion Equation was shown to have conservation laws produced by multiplying by any function, s , which is a solution of $s_t + s_{xx} = 0$.

CHAPTER FOUR

A Framework for the use of Multipliers

In this chapter a framework is developed for the use of multipliers. This framework provides a basis for the study of various aspects of conservation laws in chapter five.

In part a) the framework is defined and in parts b), c) and d) a number of well-known equations are used to illustrate it. Part e) develops the concept of an irreducible term (used in a simple form by Kruskal et al (1970)) which is important in the use of the framework.

a) The Framework

When a multiplier multiplies an equation to produce a divergence, the divergence is produced by one or both of two processes.

1. The product of the multiplier and a particular term in the equation equals a divergence by itself. This occurs in the case of the Sine-Gordon equation

$$u_{xt} - \sin u = 0. \quad (4.1)$$

Multiply the equation by the second multiplier given in chapter two, section c),

$$(2u_{xxx} + u_x^3)(u_{xt} - \sin u) = 0. \quad (4.2)$$

Each term in the equation separately forms a divergence when multiplied by the multiplier;

$$\begin{aligned}(2u_{xxx} + u_x^3)u_{xt} &= 2(u_{xx}u_{xt})_x - (u_{xx}^2)_t + 1/4 (u_x^4)_t \\ (2u_{xxx} + u_x^3)\sin u &= 2(u_{xx}\sin u)_x - (u_x^2\cos u)_x.\end{aligned}\quad (4.3)$$

2. In the product of multiplier and equation the terms which come from a particular equation term may 'interact' with those from another term to form a divergence. Consider the polynomial $(u_x u + u_{xxx})$ (that is the Korteweg-de Vries equation without its t -derivative term). Multiply this by the second Korteweg-de Vries multiplier $(u^2 + 2u_{xx})$. If the polynomial terms are considered separately, we do not get divergences

$$\begin{aligned}u_x u(u^2 + 2u_{xx}) &= 1/3 (u^3)_x + 2u_{xx}u_x u \\ u_{xxx}(u^2 + 2u_{xx}) &= u_{xxx}u^2 + (u_{xx}^2)_x.\end{aligned}\quad (4.4)$$

Adding the terms together (that is, allowing them to 'interact') does produce a divergence

$$\begin{aligned}(u_x u + u_{xxx})(u^2 + 2u_{xx}) &= 1/3 (u^3)_x + 2u_{xx}u_x u + u_{xxx}u^2 + (u_{xx}^2)_x \\ &= 1/3(u^3)_x + (u_{xx}u^2)_x + (u_{xx}^2)_x.\end{aligned}\quad (4.5)$$

In many cases both processes occur at the same time. For example, for the KdV equation consider the second multiplier $u^2 + 2u_{xx}$.

$$u_t(u^2 + 2u_{xx}) = 1/3 (u^3)_t + 2(u_x u_t)_x - (u_x^2)_t \quad (4.6)$$

and

$$(u_x u + u_{xxx})(u^2 + 2u_{xx}) = 1/3 (u^3)_x + (u_{xx}u^2)_x + (u_{xx}^2)_x. \quad (4.7)$$

The equation can be split into two parts, u_t and $u_x u + u_{xxx}$,

which separately form divergences. The terms in the second part 'interact' with each other to form a divergence.

Every evolution equation $u_t + R = 0$, R a function of u and its x -derivatives, will 'split' in this way for polynomial conservation laws. The multiplier will not contain t -derivatives as these can be eliminated by use of the equation. Thus the product of u_t and the multiplier will contain a t -derivative while the rest of the product of equation and multiplier will not. If a divergence is to be obtained u_t multiplied by the multiplier must equal a divergence.

Because of this, any evolution equation which has only one term other than u_t will have polynomial conservation laws only if the two terms can separately form divergences from the same multiplier.

For conservation laws with explicit dependence on x , the above analysis still holds as it is the t -derivative which forces the 'split'. However if explicit dependence on t is allowed, there may be interaction since the t -derivative can be 'cancelled' if it is multiplied by t ;

$$\begin{aligned} tu_t &= (tu)_t - u \\ &= (tu)_t - (xu)_x + xu_x \end{aligned} \tag{4.8}$$

$$\Rightarrow tu_t - xu_x = (tu)_t - (xu)_x. \tag{4.9}$$

Thus in this case u_t can interact with x -derivative terms.

The conservation law found by Miura et al (1968) for the KdV equation

$$\begin{aligned} d/dt(xu - 1/2 tu^2) + d/dx(1/2 xu^2 + xu_{xx} - u_x - 1/3 tu^3 \\ - tu_{xx}u + 1/2 tu_x^2) = 0 \end{aligned} \quad (4.10)$$

with multiplier $x - ut$ is an example of interaction involving u_t and terms with explicit t -dependence.

At this point a few definitions are necessary.

Total cardinality

Consider a term which involves x , t , u , its x - and t -derivatives, and x - and t -integrals of the form

$$\begin{aligned} a(x, t, \dots, u, \dots) \int_{-\infty}^x b(x', t, u, \dots) dx', \\ a(x, t, \dots, u, \dots) \int_{-\infty}^t b(x, t', u, \dots) \int_{-\infty}^x c(x', t', u, \dots) dx' dt' \\ \text{etc.} \end{aligned} \quad (4.11)$$

These integrals will be represented by the shorthand

$$\begin{aligned} a \int^x b \\ a \int^t b \int^x c \\ \text{etc.} \end{aligned} \quad (4.12)$$

Define the total cardinality to be the set of numbers

$$\begin{aligned} (\text{number of } u\text{'s, number of } x\text{'s, number of } t\text{'s,} \\ \text{number of } x\text{-integrals, number of } t\text{-integrals,} \\ \text{number of } x\text{-derivatives, number of } t\text{-derivatives}). \end{aligned} \quad (4.13)$$

Thus the term

$$x^2 t u_x \int^x u_{xt} u_t \int^t u^2 \quad (4.14)$$

has total cardinality (5, 2, 1, 1, 1, 2, 2).

In the case of polynomials in u and its x - and t -derivatives only the shorthand version

$$(3, 3, 2) \equiv (3, 0, 0, 0, 0, 3, 2) \quad (4.15)$$

will be used.

Thus $u_{xx}u_{xt}u_t$ has total cardinality

$$(3, 0, 0, 0, 0, 3, 2) \text{ or } (3, 3, 2). \quad (4.16)$$

Subrank

It is possible for terms with different total cardinalities to interact. For example

$$xu_{xxt} \quad (4.17)$$

has total cardinality $(1, 1, 0, 0, 0, 2, 1)$

and

$$u_{xt} \quad (4.18)$$

has total cardinality $(1, 0, 0, 0, 0, 1, 1)$

but

$$xu_{xxt} + u_{xt} = (xu_{xt})_x. \quad (4.19)$$

A subrank is defined as the set of total cardinalities whose terms may interact with each other to form divergences. (A term will be variously described as 'being in' or 'having' a particular subrank).

A subrank is characterised by N_u, N_x, N_t where

N_u = number of u's,

N_x = Number of x-derivatives minus number of x's minus number of
x-integrals, (4.20)

and N_t = Number of t-derivatives minus number of t's minus number
of t-integrals.

N_u , N_x and N_t are the same for all members of a particular subrank. For example

$$xu_x \int^x u_{xt} \int^t u^2 \text{ and } u^4 \quad (4.21)$$

belong to the same subrank since $N_u = 4$, $N_x = 0$ and $N_t = 0$ for both.

A subrank will be represented by

$$(N_u, N_x, N_t). \quad (4.22)$$

The generalisation to more than two independent variables is obvious.

In general the only terms with subranks having $N_u = 0$ are polynomials in x and t .

The difference between two subranks

$$(a_1, b_1, c_1) \text{ and } (a_2, b_2, c_2)$$

is expressed as the subrank

$$(a_1 - a_2, b_1 - b_2, c_1 - c_2).$$

For example the subrank difference between

$$\begin{aligned} &u_{xxx}u_tu \quad (\text{subrank } (3, 3, 1)) \\ \text{and} \quad &u_{xx}u_{tt}u \quad (\text{subrank } (3, 2, 2)) \end{aligned} \quad (4.24)$$

is $(0, 1, -1)$. It is assumed that

$$a_1 - a_2 \geq 0. \quad (4.23)$$

It is possible for a subrank difference to have N_u equal zero but the difference is not equivalent to a polynomial in x and t in any way.

In the case of polynomial terms without integrals or explicit

dependence on x or t , the subrank and the total cardinality (in its shorthand form) will be the same.

Rank

A rank is a set of subranks the elements of which differ by some multiple of a given subrank.

For example the subranks $(4, 3, 3)$, $(2, 1, 1)$ and $(1, 0, 0)$ are all members of the rank with the basic difference $(1, 1, 1)$. A rank can be defined by its basic difference and one of its members. For example, the difference $(1, 2, 1)$ and the term $(1, 0, 1)$ define the rank containing $(1, 0, 1)$, $(2, 2, 2)$, $(3, 4, 3)$, $(4, 6, 4)$ et cetera.

A choice of a difference splits the set of all possible subranks into equivalence classes - each equivalence class is a rank. Different choices produce different splittings. Thus $(2, 1, 1)$ is in the same rank as $(1, 0, 0)$ for the difference choice $(1, 1, 1)$ but not for the choice $(1, 1, 0)$.

The term 'rank' was used in a similar context by Kruskal et al (1970) in their study of the conservation laws of the KdV and Modified Korteweg-de Vries equations. Their ranks are equivalent to those defined by the differences $(1, -2, 0)$ (used for the conservation laws of the KdV equation) and $(2, -2, 0)$ (used for the Modified Korteweg-de Vries equation).

Superrank

A superrank is a generalised form of rank which allows more than one difference between subranks.

Thus for any two subranks in a superrank the difference must be a sum of the differences which occur in the superrank.

For example, if the differences in the superrank are (1, 1, 0) and (2, 1, 1) then (1, 1, 1) and (6, 4, 3) are members of the same superrank since the difference is

$$(5, 3, 2) = (1, 1, 0) + [2 \times (2, 1, 1)]. \quad (4.25)$$

As in the case of ranks, the choice of differences splits the set of subranks into equivalence classes and each class is a superrank.

Splitting

If one has an equation which has conservation laws formed by multipliers which produce divergences separately for every term of the equation (that is 'splitting' occurs) then the multipliers will have terms which are all of the same subrank. If such a multiplier had more than one subrank it could be divided into independent multipliers each containing only one subrank because terms of differing subrank cannot, by definition, add together to produce a divergence.

For example the classical wave equation

$$u_{tt} - u_{xx} = 0 \quad (4.26)$$

has a conservation law formed by the multiplier

$$u_{xxt}u_t + 1/2 u_{xx}u_{tt} + u_{xtt}u_x + 1/2 u_{xt}^2. \quad (4.27)$$

The terms of this multiplier are all of the same subrank - in fact they are all of the same total cardinality.

Each term of the equation separately forms a divergence with the multiplier

$$\begin{aligned} & u_{tt}(u_{xxt}u_t + 1/2 u_{xx}u_{tt} + u_{xtt}u_x + 1/2 u_{xt}^2) \\ &= (u_{tt}u_{xt}u_t + 1/2 u_x u_{tt}^2)_x - 1/2 (u_t u_{xt}^2)_t \\ & u_{xx}(u_{xxt}u_t + 1/2 u_{xx}u_{tt} + u_{xtt}u_x + 1/2 u_{xt}^2) \\ &= (1/2 u_{xx}^2 u_t + u_{xx}u_{xt}u_x)_t - 1/2 (u_{xt}^2 u_x)_x. \end{aligned} \quad (4.28)$$

In any search for multipliers one must remember the possibility that a multiplier of single rank may form a divergence by separately forming a divergence with each term. To cover this possibility one needs to ensure that every possible subrank is considered either by testing it or by finding conditions which exclude it from producing a divergence separately with each term of the equation. An example of the latter is the linear evolution equation case studied in chapter three. If there is an even term in a linear evolution equation no polynomial multiplier can separately form a divergence with u_t and with the even term.

Interaction

The vast bulk of conservation laws are formed by multipliers which produce interaction between the terms of the equation. In some cases (for example the polynomial conservation laws of evolution equations) the equation is split into different parts each of which separately forms a divergence. Those parts which

have more than one term will have interaction between terms.

In the product of multiplier and equation there will be terms with differing subranks. Some subranks will have terms which come from only one equation term while others will have terms from two or more equation terms. For example, consider the non-t-derivative part of the KdV equation, $u_x u + u_{xxx}$. Multiplying by

$$u^2 + 2u_{xx}$$

one gets

$$\begin{array}{ll} u_x u^3 & \text{subrank } (4, 1, 0) \\ 2u_{xx} u_x u + u_{xxx} u^2 & \text{subrank } (3, 3, 0) \\ 2u_{xxx} u_{xx} & \text{subrank } (2, 5, 0). \end{array} \quad (4.29)$$

Both equation terms contribute to the sum with subrank (3, 3, 0).

If two terms in an equation are to contribute to the same subrank the corresponding terms in the multiplier must have a difference which is the same as that between the two equation terms. Thus the difference between the terms in the equation must also occur in the multiplier. Thus for the above example the difference between $2u_{xx}$ (subrank (1, 2, 0)) and u^2 (subrank (2, 0, 0)) is the same as that between u_{xxx} (subrank (1, 3, 0)) and $u_x u$ (subrank (2, 1, 0)), that is (1, -2, 0). If this were not the case there would be no interaction.

b) Polynomial equations with two subranks

An equation with two subranks may have more than two terms since there can be more than one term per subrank.

For an equation with two subranks there is only one subrank difference and thus, if interaction is to occur this difference must appear in the multiplier. Two particular subranks in the multiplier may differ by some multiple of the equation difference but if this is the case there will be subranks between these, thus forming a 'line' of subranks in which the difference between neighbouring subranks is the equation difference. If this is not true then the multiplier will split into two or more parts, each forming a separate divergence and there will be more than one multiplier. Consider for example the non-t-derivative part of the KdV equation,

$$u_x u + u_{xxx} \quad (4.30)$$

The third multiplier for the KdV equation (with subranks below the terms) is

$$\begin{array}{ccccccc} 18/5 u_{xxxx} & + & 3u_x^2 & + & 6u_{xx}u & + & u^3. \\ (1, 4, 0) & & (2, 2, 0) & & (3, 0, 0) & & \end{array} \quad (4.31)$$

The difference between (1, 4, 0) and (2, 2, 0) and between (2, 2, 0) and (3, 0, 0) is (1, -2, 0), the same as that between the subranks of u_{xxx} (1, 3, 0) and $u_x u$ (2, 1, 0).

Thus all terms in the multiplier for a two subrank equation will be in the same rank. The rank will have the difference which

occurs in the equation.

Although this condition severely restricts the possible form of the multiplier, it is too general to provide useful conditions for the existence of conservation laws for an arbitrary equation. There are two reasons for this

1. The number of total cardinalities with the same subrank is infinite. For example the terms $x^n u_n$, $n = 0, 1, 2, \dots$ all belong to the same subrank $(1, 0, 0)$. This does not occur if one considers only polynomial multipliers; in this case the only total cardinality (in its shorthand form) is equal to the subrank.

2. The number of subranks for a given rank is infinite. For example for the rank which has the difference $(1, 1, 1)$ and which contains the subrank $(2, 3, 1)$, every subrank of the form $(2 + n, 3 + n, 1 + n)$, $n = 0, 1, 2, \dots$ will be a member of the rank. If, however, one considers the polynomial multipliers of equations with two subranks, the difference of which contains a negative element, then there is a finite number of subranks for each rank. For example, if the equation difference is $(1, -2, 0)$ and if one only considers polynomial multipliers, then the rank which contains the subrank $(1, 6, 0)$ will contain only the following subranks

$$(1, 6, 0), (2, 4, 0), (3, 2, 0), (4, 0, 0). \quad (4.32)$$

Since only polynomial multipliers are being considered, these are the only total cardinalities. Thus a polynomial multiplier having

terms of this rank will be of the following form

$$au_6 + b_1u_4u + b_2u_3u_1 + b_3u_2^2 + c_1u_2u^2 + c_2u_x^2u + du^4. \quad (4.33)$$

In order to see whether there exists a multiplier of this form, it only remains to find the conditions on the coefficients a, b_1, \dots, d under which a conservation law is produced when the equation is multiplied by this polynomial. To do this, carry out the multiplication and manipulate the product to see if a divergence can be found.

A systematic way of finding out the conditions on the coefficients of a polynomial with undetermined coefficients for that polynomial to equal a divergence is given in chapter five. This method can also be used to determine whether a polynomial with known coefficients equals a divergence.

For equations which do not have a difference with a negative element or for non-polynomial multipliers a systematic search such as the one presented in chapter five is required to find multipliers.

c) Polynomial equations with more than two subranks

In the case of equations with more than two subranks, the ideas just presented may still hold for polynomial multipliers. Two examples follow.

1. When considering polynomial multipliers, evolution equations of the form $u_t + R_1 + R_2 = 0$ (where u_t , R_1 and R_2 have different subranks) can be split into two parts, u_t and $R_1 + R_2$. Each part separately produces a divergence.

If $R_1 + R_2$ obeys the condition given above for a two subrank equation (that is, there is a negative element in the equation difference) then the polynomial multipliers of $R_1 + R_2$, if they exist, can be found. Once a multiplier has been found for $R_1 + R_2$, one need simply multiply it by u_t to ascertain whether or not it is a multiplier for the entire equation.

The KdV and MKdV equations are of the form $u_t + R_1 + R_2 = 0$ and both have negative elements in the subrank difference between R_1 and R_2 . The form of the polynomial multipliers for these equations is restricted by the need for them to form divergences with u_t separately from the rest of the equation.

Let $F = F_1 + \dots + F_n$ be a multiplier where each F_i contains all the terms with a particular subrank appearing in F . For the KdV equation the difference between F_i and F_{i+1} will be $(1, -2, 0)$, the difference between $u_x u$ and u_{xxx} , and for the MKdV the difference will be $(2, -2, 0)$, the difference between $u_x u^2$ and u_{xxx} . If F_1 is the term with the lowest number of u 's and F_n is the term with the highest then all subranks between these which differ from F_1 by a multiple of $(1, -2, 0)$ for the KdV equation or $(2, -2, 0)$ for the MKdV equation must occur.

For both equations u_{xxx} is the only linear term other than

u_t . Thus $F_1 u_{xxx}$ must equal a divergence. From chapter three one knows that if F_1 is to produce a divergence with both u_{xxx} and u_t at the same time then $F_1 = u_{2n}$, $n = 0, 1, 2, \dots$

Since F_n and $u^m u_x$ (where $m = 1$ for the KdV equation and $m = 2$ for the MKdV equation) are the terms with the highest numbers of u 's both $F_n u^m u_x$ and $F_n u_t$ must equal divergences. Thus

$$\begin{aligned} \frac{dF_n}{dt} &= \sum_{i=1}^{\infty} (-1)^i u_{ti} \frac{\delta F_n}{\delta u_i} \\ \Rightarrow \frac{dF_n}{dx} &= \sum_{i=1}^{\infty} (-1)^i u_{xi} \frac{\delta F_n}{\delta u_i} \end{aligned} \quad (4.34)$$

since F does not contain t -derivatives,

and

$$\begin{aligned} - \sum_{i=1}^{\infty} (-1)^i \frac{\delta(u^m u_x)}{\delta u_i} \frac{d^i F_n}{dx^i} &= \sum_{i=1}^{\infty} (-1)^i (u^m u_x)_i \frac{\delta F_n}{\delta u_i} \\ \Rightarrow u^m \frac{dF_n}{dx} &= \sum_{i=1}^{\infty} (-1)^i (u^m u_x)_i \frac{\delta F_n}{\delta u_i}. \end{aligned} \quad (4.35)$$

From (4.34) one finds

$$\begin{aligned} u^m \frac{dF_n}{dx} &= u^m \sum_{i=1}^{\infty} (-1)^i u_{xi} \frac{\delta F_n}{\delta u_i} \\ &= \sum_{i=1}^{\infty} (-1)^i u^m u_{xi} \frac{\delta F_n}{\delta u_i}. \end{aligned} \quad (4.36)$$

Comparing (4.35) and (4.36) one sees that $\delta F_n / \delta u_i$ equals zero only if $i = 0$. This implies that $F_n = u^j$, $j = 0, 1, \dots$

Thus

$$F = u_{2n} + \dots + u^{n+1} \quad (4.37)$$

for the KdV equation and

$$F = u_{2n} + \dots + u^{2n+1} \quad (4.38)$$

for the MKdV equation. The number of u 's in the final terms is found by calculating the number of subranks required between u_{2m} and u^j .

Kruskal et al (1970) used their restricted form of rank for the KdV and MKdV equations. Their conclusions are equivalent to those found above - only those ranks containing u_{2n} could produce multipliers. Those containing u_{2n+1} could not.

The equations

$$u_t + u^m u_x + u_{2p+1}, \quad m > 2, \quad p \gg 1, \quad (4.39)$$

are also evolution equations of the form under discussion in this section. These are well-known [Miura (1974)] to have only three polynomial conservation laws (the equation itself plus those produced by the multipliers u and $(m+1)u_{2p} + u^{m+1}$) and appear to have no others.

Consider the equation

$$u_t + u^3 u_x + u_{xxx} = 0. \quad (4.40)$$

The restrictions on the form of its multipliers will be the same as on the KdV and MKdV equations. Thus the multipliers must be of the form $u_{2n} + \dots + u^{3n+1}$.

The multiplier of next highest rank beyond those which are known to give conservation laws would be

$$u^7 + au^3 u_{xx} + bu_x^2 u^2 + cu_{xxxx}. \quad (4.41)$$

Multiplying this by the equation, one obtains the following two conditions

$$u_t(a u_{xx}^3 + b u_x^2 u^2) = \text{divergence} \quad (4.42)$$

($u_t u^7$ and $u_t u_{xxxx}$ are divergences by themselves)

$$\begin{aligned} \Rightarrow (b - 3a/2) u_x^2 u^2 u_t &= \text{divergence} \\ \Rightarrow b - 3a/2 &= 0 \end{aligned} \quad (4.43)$$

and

$$\begin{aligned} a u_{xx}^6 u_x + b u_x^3 u^5 + c u_x^3 u_{xxxx} \\ + u^7 u_{xxx} + a u_{xxx} u_{xx} u^3 + b u_{xxx} u_x^2 u^2 &= \text{divergence} \end{aligned} \quad (4.44)$$

($u^{10} u_x$ and $u_{xxxx} u_{xxx}$ are divergences by themselves)

$$\begin{aligned} (b - 6(a-7)/2) u_x^3 u^5 \\ \Rightarrow - (2(b-3c) + 3(a-c)/2) u_{xx}^2 u_x u^2 \\ + (b-3c)/2 u_x^5 &= \text{divergence} \end{aligned} \quad (4.45)$$

$$b - 3a + 21 = 0,$$

$$\Rightarrow 2b - 15/2 c + 3/2 a = 0 \quad (4.46)$$

$$\text{and } b - 3c = 0.$$

A quick check shows that no solution exists for these simultaneous equations (that is for the three equations (4.46) and equation (4.43)). Thus, for this rank, there is no multiplier. The KdV and MKdV equations have multipliers for their ranks which correspond to this one. Thus there is a stark contrast between the KdV and MKdV equations on the one hand and the equation

$$u_t + u_x u^3 + u_{xxx} = 0, \quad (4.47)$$

on the other.

Even as low as the fourth rank there is no polynomial

multiplier for this equation whereas there are infinite numbers of polynomial multipliers for the KdV and MKdV equations.

Since it is known that equations of the form

$$u_t + u^m u_x + u_{2p+1} = 0, \quad m > 2, \quad p \geq 1,$$

have only three conservation laws one would expect a similar result to that just found for all such equations.

If one considers non-polynomial multipliers (for example those containing x and t explicitly) the special results for evolution equations of three subranks no longer apply since u_t can no longer be treated as a separate part of the equation. For example, the KdV equation has the multiplier $x - ut$. This has subranks $(0, -1, 0)$ and $(1, 0, -1)$. The difference between these is $(1, 1, -1)$, the same as that between u_t and $u_x u$. In this case, it is u_{xxx} which forms a divergence separately.

2. If all the differences between the subranks of an equation are the same, or are at most a multiple of a particular difference, then the form of the multipliers is restricted in the same way as for two subrank equations with the exception that the differences in the multiplier may be up to - for an n subrank equation - n times the basic equation difference. For example if one has an equation

$$a + b + c = 0 \tag{4.48}$$

where the subrank difference between a and b and between b and c is $(d_1, d_2, 3)$, then

$$e + f \quad (4.49)$$

where the subrank difference between e and f is $2 \times (d_1, d_2, d_3)$, will be a multiplier if ae , be , bf , and $af + ce$ all separately equal divergences. Thus one gets interaction (for $af + ce$) even though the basic equation difference, (d_1, d_2, d_3) , does not appear in the multiplier. For an evolution equation this condition applies to the non- t -derivative part of the equation with a restriction to polynomial multipliers.

As in the case of two subrank equations, if there is a negative element in the equation difference then there will be a finite number of possible subranks for each rank. This occurs for the hierarchy of higher-order KdV equations [Caudrey, Dodd and Gibbon (1976)]

$$\begin{aligned} u_t + (u_4 + 30uu_2 + 60u^3)_x &= 0 \\ u_t + (u_6 + 56uu_4 + 70u_2^2 + 840u^2u_2 + 840u^4)_x &= 0 \\ u_t + (u_8 + 90uu_6 + 420u_2u_4 + 2520u^2u_4 + 6300u_2^2u &+ 25200u^3u_2 + 15120u^5)_x = 0 \end{aligned} \quad (4.50)$$

and so on. Each subrank in these equations is often represented by more than one term. The equation difference in each case is $(1, -2, 0)$, the same as for the KdV equation.

If an equation has more than one difference between its subranks and if its terms do not divide into parts which obey the rules above (each part separately forming a divergence with the multiplier) then one obtains a 'network' multiplier instead of a 'line' multiplier. A 'network' multiplier contains terms which

are all in the superrank which has the same subrank differences as the equation. For example the subranks

$$(1, 1, 1) - (2, 2, 2) - (3, 3, 3) - (4, 4, 4) \quad (4.51)$$

form a line multiplier while

$$\begin{array}{ccccc}
 & (1, 2, 0) & & (3, 3, 3) & \\
 & \swarrow \quad \searrow & & \swarrow \quad \searrow & \\
 (2, 3, 1) & & (2, 2, 2) & & (4, 3, 5) \\
 & \swarrow \quad \searrow & & \swarrow \quad \searrow & \\
 & (1, 1, 1) & & (3, 2, 4) &
 \end{array} \quad (4.52)$$

- this has differences $(1, 1, 1)$ and $(1, 0, 2)$ - is an example of a network multiplier. An equation which requires network multipliers is

$$w_t + (w + \frac{k^2}{6} w^2) w_x + w_{xxx} = 0, \quad k \text{ constant.} \quad (4.53)$$

This equation has the differences $(1, 0, 0)$, $(1, -2, 0)$ and $(2, -2, 0)$ in its non- t -derivative part and it has been shown to have an infinite number of polynomial conservation laws by Kruskal (1975). It is closely related to the KdV and MKdV equations - it tends towards the KdV equation as k tends to zero and to the MKdV equation as k tends towards infinity.

A simple method of constructing network multipliers is to start with a 'basic subrank' and then add to this all the subrank differences that appear in the equation or equations. The result of this will be a set of subranks (including the basic subrank) which has the same structure as the original equation (as the set of subranks in a linear multiplier has the same structure as the equation). One could study a range of multipliers by starting

with different 'basic subranks'. An example of how this structure can produce constants of the motion is given in chapter five, section c. More complex multipliers could be produced by taking the above structure and adding each subrank difference of the equation to each subrank difference in the simple multiplier. This procedure could be repeated to produce ever more complex multipliers.

d) Non-polynomial equations

The concepts of subrank and rank are useful ones in the study of conservation laws and it is desirable that their use be extended to non-polynomial equations. This can certainly be done but since more complex functions are involved, care must be taken. This section gives a few examples of the problems that can arise and how they might be resolved.

If one considers the multipliers given for the Sine-Gordon equation in chapter two, one sees that the differences between the subranks is $(2, 0, 0)$ - equivalent to the term u^2 . This is unexpected since it was stated in a previous section that if terms in an equation separately form divergences with the multiplier - as is the case with the Sine-Gordon equation the multiplier will have only one subrank.

This apparent anomaly could be resolved in either of the following two ways (other possibilities may also exist).

1. One may write $\sin u$ as a series

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \dots \quad (4.54)$$

leading to an equation with an infinite number of terms. It is now possible to calculate the subrank of each term whereas the subrank has not been defined for $\sin u$. This series has a difference of (2, 0, 0) between its terms - the same as that between the terms of the multiplier.

2. Alternatively one can try to define a subrank directly for $\sin u$. Consider

$$\begin{aligned} (\sin u)_{xx} &= (u_x \cos u)_x \\ &= u_{xx} \cos u - u_x^2 \sin u. \end{aligned} \quad (4.55)$$

For each term the number of x -derivatives is the same but the number of factors of u which are multiplied by a trigonometric function is zero, one or two. Since the basic concept of a subrank is that its terms can combine to form divergences and since the terms in the second line of the above equation differ in the number of factors of u , one must allow for variations in the number of u 's within a subrank if one is to define a subrank for $\sin u$. (Previously a differing number of factors of u occurred only for ranks.)

Comparing the two occurrences of $\sin u$ in the above equation one can see that the two terms differ by two in the number of u 's. Thus $\sin u$ can be regarded as having subranks which differ by (2, 0, 0) - the subrank difference of the multipliers of the

Sine-Gordon equation.

Although the subrank of $\sin u$ has not been precisely defined here these two procedures show that the subrank difference $(2, 0, 0)$ will occur for the Sine-Gordon equation.

It has already been noted in chapter two that the multipliers of the Sine-Gordon and Modified Korteweg-de Vries equations are related by the transformation

$$u \rightarrow \sqrt{3/2} u_x. \quad (4.56)$$

If one makes this substitution in the Modified Korteweg de Vries equation one gets

$$3/2 (u_{xt} + 3/2 u_{xx} u_x^2 + u_{xxxx}) = 0. \quad (4.57)$$

The subrank difference between the last two terms is $(2, 0, 0)$ - the same as for the Sine-Gordon equation. One would expect the same subrank difference to occur in both equations as the multipliers are, to within the above transformation, the same and thus the differences between terms in the equation which interact must also be the same.

With the Modified Korteweg-de Vries equation in its transformed form, both equations have the term u_{xt} . Since the multipliers are polynomials in the x -derivatives of u (as the transformation does not involve t -derivatives this will also be the case for the MKdV equation) the term u_{xt} will place the same restriction on the multipliers for both equations. In both cases u_{xt} will form a divergence by itself.

Thus since the basic difference for the non- t -derivative

terms is the same for both equations and since the equations have the same term containing a t -derivative, it is not altogether surprising that the multipliers for the two equations are closely related.

One also has a problem defining subranks for the Toda differential-difference equation. Since one has

$$d/dt (e^{-(u_n - u_{n-1})}) = -(u_{n,t} - u_{n-1,t}) e^{-(u_n - u_{n-1})} \quad (4.58)$$

how does one define a subrank for the exponential terms?

This too can be resolved by writing the exponential term as an infinite series

$$e^{-(u_n - u_{n-1})} = 1 - (u_n - u_{n-1}) + 1/2 (u_n - u_{n-1})^2 - \dots \quad (4.59)$$

The most obvious way to define the subrank is to require that the subrank of every term in $(u_n - u_{n-1})^m$ be the same. Thus the terms in $(u_n - u_{n-1})^2$, that is, u_n^2 , $u_n u_{n-1}$, u_{n-1}^2 , all have the same subrank. This can be done by defining N_u (the first element in a subrank) to be the total number of u_i in a term regardless of the values of i . Thus u_n^2 , $u_n u_{n-1}$ and u_{n-1}^2 all have the same subrank $(2, 0, 0)$. Similarly the term $u_{n,tt}$ in the Toda equation has subrank $(1, 0, 2)$.

If one considers the third multiplier for the Toda equation (2.79)

$$u_{n,t}^2 - e^{-(u_n - u_{n-1})} + e^{-(u_{n+1} - u_n)} \quad (4.60)$$

one can see that the subranks in this multiplier are $(2, 0, 2)$, $(1, 0, 0)$, $(2, 0, 0)$ et cetera. Thus there are more

than one basic equation difference. This arises because the t -derivative term in the equation, $u_{n,tt}$, interacts with the infinite number of other terms. This can occur since the Toda equation is not an evolution equation and thus its polynomial multipliers - polynomial, that is, in the u_i and their derivatives- may contain t -derivatives.

Another equation studied in this thesis which requires special attention is the Benjamin-Ono equation. This equation involves the integral operator H where

$$Hq(x) = \int \frac{q(z)}{z-x} dz. \quad (4.61)$$

For this equation an extra element, N_H , would be needed in the subrank to keep account of the number of H 's. Thus, if the subrank is now of the form

$$(N_q, N_x, N_t, N_H) \quad (4.62)$$

then the difference between the two non- t -derivative terms

$$2qq_x + Hq_{xx} \quad (4.63)$$

is $(1, -1, 0, -1)$. However, one also has

$$H^2 f = -f. \quad (4.64)$$

Thus terms such as qq_{xx} and $(Hq)(Hq_{xx})$ would be of the same subrank. N_H will therefore give the maximum number of H 's. If N_H is even the possible number of H 's will be $0, 2, 4, \dots, N_H$ while if N_H is odd the possibilities will be $1, 3, 5, \dots, N_H$. Thus, in the multiplier

$$4q^3 + 6qH(q_x) - 6qq_xH - 4q_{xx} \quad (4.65)$$

where one would expect the final term to have two H's, there are no H's.

The concept of a subrank can be successfully used for non-polynomial equations but care is needed to ensure that the subranks are defined correctly.

e) Irreducible Terms

In order to calculate coefficients a concept used by Kruskal et al (1970), that of irreducible terms, can be generalised. Kruskal et al consider only those irreducible terms involving u and its x -derivatives. The generalisation used here includes t -derivatives and explicit x and t dependence as well as the x - and t -integrals in the form given in part a.

The definition of an irreducible term is best given by an example. Consider the term $u_{xxxx}u_xu$.

$$\begin{aligned}
 u_{xxxx}u_xu &= (u_{xxx}u_xu)_x - u_{xxx}u_{xx}u - u_{xxx}u_x^2 \\
 &= (u_{xxx}u_xu)_x - 1/2 (u_{xx}^2u)_x - (u_{xx}u_x^2)_x \quad (4.66) \\
 &\quad + 1/2 u_{xx}^2u_x + 2u_{xx}^2u_x \\
 &= (u_{xxx}u_xu - 1/2 u_{xx}^2u - u_{xx}u_x^2)_x \\
 &\quad + 5/2 u_{xx}^2u_x.
 \end{aligned}$$

If any attempt is made to further manipulate the term $u_{xx}^2u_x$ one inevitably returns to one of the earlier terms. For example

$$u_{xx}^2u_x = (u_{xx}^2u)_x - 2u_{xxx}u_{xx}u \quad (4.67)$$

or

$$u_{xx}^2 u_x = 1/2 (u_{xx} u_x^2)_x - 1/2 u_{xxx} u_x^2. \quad (4.68)$$

Thus the term $u_{xx}^2 u_x$ is 'irreducible'.

In the case of polynomials in u and its x -derivatives (the case considered by Kruskal et al) the definition of the set of irreducible terms is simple; a term is irreducible if its highest order factor is at least squared. Any term for which this is not the case can always be reduced by taking an x -derivative from the highest order factor and placing it on lower order factors. In the case of a term such as $u_{n+1} u_n u_m$, $n > m$ one gets

$$u_{n+1} u_n u_m = 1/2 (u_n^2 u_m)_x - 1/2 u_n^2 u_{m+1}. \quad (4.69)$$

Before giving the rules for the irreducibility of more complex terms it is necessary to establish conventions for writing terms. If conventions are not established the irreducibility of a term could depend on the way in which it was written since some of the rules for irreducibility depend on the positions of particular factors.

The conventions are

1. Factors with the highest number of x -derivatives are written on the left, those with the lowest on the right. For example

$$u_{xxx} u_{xx}^2 u_x u^2. \quad (4.70)$$

2. If two factors have the same number of x -derivatives the one with the highest number of t -derivatives is written on the left.

For example

$$u_{xxx} u_{xxtt} u_{xxt} u_x u_{ttt}. \quad (4.71)$$

If more than two independent variables are used this system can be extended. Thus one gets

$$u_{xxx}u_{xxtt}u_{xxtt}u_xu_{yy}. \quad (4.72)$$

Similarly, for more than one dependent variable one chooses a particular ordering of the dependent variables. Thus

$$u_{xx}u_xv_{xx}v_x. \quad (4.73)$$

If integrals are included then the parts separated by the integrals must each separately obey these conventions. For example

$$u_{xt}u_x \int^x u_{xxt}u_{tt} \int^t u_{xxx}u_{xt} \int^x u_xu_{ttt}u_t. \quad (4.74)$$

Rules for irreducibility will be given for the case of one dependent variable, u , and two independent variables, x and t . Initially the rules for polynomials in u and its derivatives will be given. These rules will then be extended to include explicit functions of x and t and integrals of the form used in part a.

The case of two dependent variables and one independent variable will also be considered. Although rules have not been developed for more than two dependent or two independent variables such rules should be of the same form - just more complicated!

Two independent variables, one dependent variable

Let $U(X_H)$ be the left hand factor in a term which obeys the convention for the ordering of factors given above. Here X_H is the number of x -derivatives in $U(X_H)$. The factor immediately to the right $U(X_H)$, denoted by $U(X_N)$, will have X_N x -derivatives

where X_N is less than or equal to X_H .

$$\text{Define } D(X) = X_H - X_N \geq 0. \quad (4.75a)$$

For terms in one independent variable and one dependent variable, the condition for irreducibility is simply $D(X) = 0$. This was the definition of irreducibility used by Kruskal et al (1970).

Let $U(T_H)$ be the factor furthest to the left with the highest number of t -derivatives. If there are other factors with the same number, T_H , of t -derivatives then denote the one nearest to $U(T_H)$ by $U(T_N)$ where $T_N = T_H$, otherwise let $U(T_N)$ be the factor furthest to the left with the next highest value of $T_N < T_H$.

$$\text{Define } D(T) = T_H - T_N \geq 0. \quad (4.75b)$$

As in the case of one independent variable given above ($D(X) = 0$ was the requirement for irreducibility) the values of $D(T)$ and $D(X)$ must be reduced as much as possible. For $D(T)$ or $D(X)$ greater than or equal to two it is always possible to perform a manipulation to reduce the value to one. For example

$$\begin{aligned} u_{xxx}u_{xtt}u^2 &= (u_{xxt}u_{xtt}u^2)_x \\ &\quad - u_{xxtt}u_{xxt}u^2 \\ &\quad - 2u_{xxt}u_{xtt}u_xu. \end{aligned} \quad (4.76)$$

It is not, however, always possible to reduce terms with $D(X) = 1$ or $D(T) = 1$ without eventually reproducing the term which one is trying to reduce. For example

$$\begin{aligned}
u_{xxx}u_{xxt}u_{xx}u_{tt} &= (u_{xxt}u_{xxt}u_{xx}u_{tt})_x \\
&- u_{xxx}u_{xxt}u_{xx}u_{tt} \\
&- u_{xxx}u_{xxt}u_{xxt}u_{tt} \\
&- u_{xxt}u_{xxt}u_{xx}u_{xt}.
\end{aligned} \tag{4.77}$$

Manipulating the second non-derivative term on the right-hand side one gets

$$\begin{aligned}
- u_{xxx}u_{xxt}u_{xxt}u_{tt} &= - (u_{xxt}u_{xxt}u_{xx}u_{tt})_x \\
&+ u_{xxx}u_{xxt}u_{xx}u_{tt} \\
&+ u_{xxx}u_{xxt}u_{xx}u_{tt} \\
&+ u_{xxt}u_{xxt}u_{xx}u_{xt}.
\end{aligned} \tag{4.78}$$

The second non-derivative term on the right-hand side is the same as the original term. It also has the same sign and therefore it cannot simply be added to the left-hand side. Thus it is necessary to specify arbitrarily one of

$$u_{xxx}u_{xxt}u_{xx}u_{tt} \tag{4.79}$$

or

$$u_{xxx}u_{xxt}u_{xxt}u_{tt} \tag{4.80}$$

to be irreducible - here $u_{xxx}u_{xxt}u_{xx}u_{tt}$ is chosen to be irreducible (see rule 2 below).

Hence an irreducible term will have $D(X)$ and $D(T)$ equal to zero or one. The four cases ($D(X) = D(T) = 0$; $D(X) = 1, D(T) = 0$; $D(X) = 0, D(T) = 1$; $D(X) = D(T) = 1$) will be treated separately.

1. $D(T) = D(X) = 0$.

The term is irreducible since any manipulation will increase one of $D(X)$ or $D(T)$.

2. $D(X) = 1, D(T) = 0$.

The term is irreducible

(i) if the number of t -derivatives on $U(X_H)$ is greater than that on $U(X_N)$

or (ii) if there is a factor with X_N x -derivatives but fewer t -derivatives than $U(X_H)$.

For example,

$$u_{xxxxtt}u_{xxt}u_{xx}u_{tt} \quad (4.81)$$

is irreducible since the number of t -derivatives on u_{xxxxtt} is greater than that on u_{xxt} . On the other hand, consider the terms studied above,

$$u_{xxxxt}u_{xxtt}u_{xx}u_{tt} \quad (4.82)$$

and $u_{xxx}u_{xxtt}u_{xxt}u_{tt}$.

Both have fewer t -derivatives on $U(X_H)$ than on $U(X_N)$ yet it was seen that one must choose one of them to be irreducible. The second part of the rule chooses $u_{xxxxt}u_{xxtt}u_{xx}u_{tt}$ to be irreducible.

3. $D(X) = 0, D(T) = 1$.

The term is irreducible

(i) if the number of x -derivatives on $U(T_H)$ is greater than that on $U(T_N)$

or (ii) if there is a factor with T_N t -derivatives but fewer x -derivatives than $U(T_H)$.

Thus

$$u_{xxxxttt}u_{xxx}u_{xtt}u \quad (4.83)$$

which obeys (i) and

$$u_{xxx}^2 u_{xxttt} u_{xxtt} u_{xtt} \quad (4.84)$$

which obeys (ii) are irreducible.

4. $D(X) = D(T) = 1$.

The term is irreducible

(a) if

(i) the number of t-derivatives on $U(X_H)$ is greater than the number on $U(X_N)$

or (ii) there is a factor with X_N x-derivatives but fewer t-derivatives than $U(X_H)$

and

(b) if

(i) the number of x-derivatives on $U(T_H)$ is greater than the number on $U(T_N)$

or (ii) there is a factor with T_N t-derivatives but fewer x-derivatives than $U(T_H)$.

Thus

$$u_{xxtt} u_{xt}^2 u_x u \quad (4.85)$$

which obeys a(i) and b(i)

$$u_{xxt} u_{xtt} u_x u_t \quad (4.86)$$

which obeys a(ii) and b(ii)

$$u_{xxx} u_{xxtt} u_{xx} u_{xtttt} u_{ttt} \quad (4.87)$$

which obeys a(ii) and b(i) and

$$u_{xxx} u_{xtt} u_{xxt} u_{xttt} u_{tt} \quad (4.88)$$

which obeys a(i) and b(ii) are all irreducible.

When a term is reducible it must be manipulated in order to produce irreducible terms. In doing this, derivatives are 'moved'; for example, for the manipulation

$$u_{xxxt}u_{xt} = (u_{xxt}u_{xt})_x - u_{xxt}^2 \quad (4.89)$$

the x-derivative is moved. The derivative which needs to be moved is determined by the rule which applies to it.

Thus if a term with $D(X) = 1$ and $D(T) = 0$ does not obey rule two then an x-derivative must be moved (since $D(T) = 0$ and thus no t-derivative can be moved without producing terms for which $D(T)$ is greater than zero). Similarly if a term with $D(X) = 0$ and $D(T) = 1$ disobeys rule three, then a t-derivative must be moved (since $D(X) = 0$). If a term with $D(X) = D(T) = 1$ disobeys part (a) of rule four then a t-derivative must be moved and if it disobeys part (b) then an x-derivative must be moved. In the case of rule four it is possible that both x- and t-derivatives could be moved - that is, the term could disobey both (a) and (b). For example, the term

$$u_{xxxt}u_{xxtt}u_{xtt}u_{ttt} \quad (4.90)$$

requires both x- and t-derivatives to be moved.

$$\begin{aligned} u_{xxxt}u_{xxtt}u_{xtt}u_{ttt} &= (u_{xxtt}u_{xxt}u_{xtt}u_{ttt})_x \\ &\quad - u_{xxxtt}u_{xxt}u_{xtt}u_{ttt} - \dots \\ &= (u_{xxtt}u_{xxt}u_{xtt}u_{ttt})_x \quad (4.91) \\ &\quad - (u_{xxxtt}u_{xxt}u_{xtt}u_{tt})_t \\ &\quad + u_{xxxtt}u_{xxt}u_{xttt}u_{tt} + \dots \end{aligned}$$

The final term is one of the examples of an irreducible term

given for rule four.

Explicit x and t

If x and t are allowed to appear explicitly then the rules remain the same - in determining whether a term is irreducible or not one need not consider the presence of the x's and t's. Thus the form of the irreducible terms will be the same except that x's and t's will be present. However, the manipulation of a term will produce additional terms. For example

$$\begin{aligned} u_{xxxx}u &= (u_{xxx}u)_x - u_{xxx}u_x \\ &= (u_{xxx}u - u_{xx}u_x)_x + u_{xx}^2 \end{aligned} \quad (4.92)$$

$$\begin{aligned} \text{but } x^2u_{xxxx}u &= (x^2u_{xxx}u)_x - x^2u_{xxx}u_x - 2xu_{xxx}u \\ &= (x^2u_{xxx}u - x^2u_{xx}u_x - 2xu_{xx}u)_x \\ &\quad + 4xu_{xx}u_x + x^2u_{xx}^2 + 2u_{xx}u \\ &= (x^2u_{xxx}u - x^2u_{xx}u_x - 2xu_{xx}u \\ &\quad + 2xu_x^2 + 2u_xu)_x \\ &\quad + x^2u_{xx}^2 - 4u_x^2. \end{aligned} \quad (4.93)$$

Since u_{xx}^2 is irreducible so is $x^2u_{xx}^2$.

Integrals

When x- and t- integrals are included, a term must be considered in a number of parts.

The first part is that to the left of the left-most integral. For example for

$$u_{xxt}u_{tt} \int^x u_{xxx} \quad (4.94)$$

one considers first the part $u_{xxt}u_{tt}$. The first part is tested using the rules given above. If it is reducible and if the

integral to the right (an x-integral in the above example) is the same as the derivative which must be moved (an x-derivative in the above example) then the term is manipulated. Thus one gets

$$\begin{aligned} u_{xxt}u_{tt} \int^x u_{xxx} &= (u_{xt}u_{tt} \int^x u_{xxx})_x \\ &- u_{xtt}u_{xt} \int^x u_{xxx} \\ &- u_{xxx}u_{xt}u_{tt}. \end{aligned} \quad (4.95)$$

The derivative which is moved is always from the part to the left of the integral even if a higher order derivative is to the right of the integral. In this example u_{xxx} has more x-derivatives than u_{xxt} , yet the derivative which is moved is from u_{xxt} which is the highest term to the left of the integral.

Once the manipulation has been performed one starts anew with each term produced.

If the first part of the term is irreducible or if it is reducible but the derivative to be moved and the integral are with respect to different variables (for example, in the term

$$u_{xtt}u_{xt} \int^x u_{xxx} \quad (4.96)$$

the part $u_{xtt}u_{xt}$ is reducible - in fact it is a divergence - and a t-derivative must be moved but the integral is with respect to x) then one considers the next part of the term. The next part is that to the left of the next integral - this part is considered without the left-most integral. For example, the first part of the term

$$u_{xxt}u_x \int^x u_{xx} \int^t u_x \quad (4.97)$$

is irreducible. Thus the second part

$$u_{xxt}u_xu_{xx} \quad (4.98)$$

which becomes

$$u_{xxt}u_{xx}u_x \quad (4.99)$$

(to obey the writing conventions) must now be considered in the same way as the first part. This process is continued until every such part (that is every part which is to the left of some integral) has been tested. The entire term is then tested without its integrals. For example, for the term

$$\text{one tests } u_{xtt}u_{xt} \int^x u_{xxx} \quad (4.100)$$

$$u_{xxx}u_{xtt}u_{xt}. \quad (4.101)$$

If the whole term without integrals is reducible then the term with integrals is manipulated - if the term without integrals is irreducible then so is the term with integrals.

Since differentiation and integration (with respect to the same or a different variable) commute, the derivatives can be moved without regard to the integrals. For example,

$$\begin{aligned} u_{xt}u \int^x u_{xxx} &= (u_{xt}u \int^x u_{xx})_x \\ &- u_{xxt}u \int^x u_{xx} \\ &- u_{xt}u_x \int^x u_{xx}. \end{aligned} \quad (4.102)$$

If, however, the derivative to be moved is outside an integral with respect to the same variable then the integral will be eliminated in some terms. For example

$$u_{xxt}u \int^x u_t = (u_{xt}u \int^x u_t)_x - u_{xt}u_x \int^x u_t - u_{xt}u_t u. \quad (4.103)$$

Each new term that is produced by a manipulation is tested in this way and the process continues until every term is

irreducible.

This procedure is necessary in order to ensure that all possible integrals have been eliminated. For example

$$u_{xx}u_x \int^x u_{xxt} = (1/2 u_x^2 \int^x u_{xxt})_x - 1/2 u_{xxt}u_x^2 \quad (4.104)$$

(the terms $u_{xx}u_x \int^x u_{xxt}$ and $u_{xxt}u_x^2$ both have the same subrank (3, 5, 0)).

If the above procedure was not carried out then one would need to say that $u_{xx}u_x \int^x u_{xxt}$ was irreducible since no derivatives can be moved without increasing $D(X)$. By considering the part $u_{xx}u_x$ separately and finding that it is reducible one reduces the term. Thus one finds that the sum

$$u_{xx}u_x \int^x u_{xxt} + 1/2 u_{xxt}u_x^2 \quad (4.105)$$

is a divergence.

The presence of integrals prevents a term being equal to a divergence by itself unless it can be reduced to a form without integrals. For example although the term

$$u_{tt}u_t^2 = 1/3 (u_t^3)_t \quad (4.106)$$

is a divergence, the same term with an integral added

$$u_{tt}u_t \int^x u_t \quad (4.107)$$

is irreducible.

If, on the other hand, the integral or integrals in a term can be eliminated, it may equal a divergence by itself. For example

$$\begin{aligned} u_{ttt} \int^t u_t^2 &= (u_{tt} \int^t u_t^2)_t - u_{tt}u_t^2 \\ &= (u_{tt} \int^t u_t^2 - 1/3 u_t^3)_t. \end{aligned} \quad (4.108)$$

If x's and t's appear explicitly in a term containing integrals the procedure is the same. For example, the first part of

$$x^2 u_{xt} u \int x_t^2 u_{xt} \quad (4.109)$$

is $x^2 u_{xt} u$. This is irreducible so one considers $x^2 t^2 u_{xt}^2 u$. This too is irreducible - thus the entire term with integrals is irreducible.

Two dependent variables, one independent variable

Consider terms containing two dependent variables, X and Y, which are both functions of only one independent variable t. (This would occur, for example, in a system of ordinary differential equations.) The writing convention chosen here is to put all the X terms on the left and for the highest order derivatives in each variable to be on the left of its group. For example,

$$X_{tt} XY_{ttt} Y_{tt} Y.$$

A term will be irreducible if either of the following conditions is satisfied.

1. The two factors with the highest numbers of derivatives both have the same number of derivatives. For example,

$$X_{tt} XY_{tt} Y \text{ and } X_{tt}^2 XY_t Y. \quad (4.110)$$

2. The number of derivatives on the X-factor which contains the highest number of derivatives has one more derivative than the Y factor which contains the highest number. For example,

$$X_{tt} XY_t Y. \quad (4.111)$$

The second condition implies the convention that the factor with the highest number of derivatives must be an X. Thus

$$\begin{aligned} X_t XY_{tt} Y &= (X_t XY_t Y)_t \\ &- X_{tt} XY_t Y - X_t^2 Y_t Y - X_t XY_t^2. \end{aligned} \quad (4.112)$$

$X_t^2 Y_t Y$ and $X_t XY_t^2$ are both irreducible by condition one. One must choose one of $X_t XY_{tt} Y$ or $X_{tt} XY_t Y$ to be irreducible. Condition two chooses $X_{tt} XY_t Y$ - this is in line with the conventions established above for two independent variables.

The usefulness of irreducible terms lies in the property that every sum of terms can be reduced to a divergence plus a sum of irreducible terms. Since by definition a sum of irreducible terms can never equal a divergence, a simple test to find out if a particular polynomial is equal to a divergence is to reduce it to a divergence plus a sum of irreducible terms. If the result contains irreducible terms then the polynomial is not equal to a divergence.

If one considers a polynomial with arbitrary coefficients then one will obtain a sum of irreducible terms whose coefficients are sums of the arbitrary coefficients of the polynomial.

Thus one obtains conditions on the coefficients of the polynomial for it to equal a divergence - each coefficient of an irreducible term must be zero. For example, consider the most general polynomial with subrank (3, 3, 0)

$$a u_{xxx} u^2 + b u_{xx} u_x u + c u_x^3. \quad (4.113)$$

This can be manipulated as follows

$$\begin{aligned}
 & au_{xxx}u^2 + bu_{xx}u_xu + cu_x^3 \\
 &= (au_{xx}u^2 + (b-2a)/2 u_x^2u)_x \\
 &\quad + (c - b/2 + a) u_x^3.
 \end{aligned}
 \tag{4.114}$$

Thus for the polynomial to equal a divergence the condition

$$c - b/2 + a = 0 \tag{4.115}$$

must be satisfied. For example, this condition is satisfied by

$$a = b = 1 \text{ and } c = -1/2$$

and one has

$$\begin{aligned}
 & u_{xxx}u^2 + u_{xx}u_xu - 1/2 u_x^3 \\
 &= (u_{xx}u^2 - 1/2 u_x^2u)_x.
 \end{aligned}
 \tag{4.116}$$

CHAPTER FIVE

Using the New Framework

The ideas presented in chapter four are used to study conservation laws.

In part a) a slight extension is made to the result in chapter three for the equation $u_t + u_n + R = 0$. Parts b) and c) reproduce well-known results for recursion operators and the Henon-Heiles system but in a much simpler and more direct way than other methods. In part d) an algorithm for finding new conservation laws using the ideas of the new framework is presented.

a) $u_t + u_n + R = 0$

In chapter three it was shown that for R a polynomial this equation has no polynomial conserved densities unless n is odd.

Here the result will be generalised to non-polynomial R ; R will now be a function of x , u and its x -derivatives (if R contains t or t -derivatives of u then terms in R could interact with u_t and thus the proof in chapter three would no longer hold). It will be assumed, without loss of generality, that R contains no linear terms.

The essence of the proof in chapter three was that the term

or sum of terms in the multiplier which had the lowest number of u 's would need to form a divergence with u_n separately. Thus the terms in the multiplier would need to be linear and, in order that these terms also formed a divergence with u_t , u_n would need to be odd.

The result can be extended to non-polynomial R if it is noted that the crucial aspect of the argument is that no terms in R can interact with u_n when they are being multiplied by the lowest order terms in the multiplier. This will always be true so long as R does not contain any terms which have the same subrank as u_n .

If one can define a subrank for all the terms in R then the theorem of chapter three can be generalized in the following way.

There will be no polynomial conservation laws (other than possibly the equation itself) for the equation

$$u_t + u_n + R = 0 \quad (5.1)$$

where n is even and R does not contain any terms with the same subrank as u_n .

If R does contain terms with the same subrank then interaction can occur and conservation laws may be possible. For example, consider the equation

$$u_t + \frac{u^2}{u} + u_{xx} = 0. \quad (5.2)$$

If one considers $1/u$ to contribute -1 to N_u , then the second term has a subrank of $(1, 2, 0)$, the same as u_{xx} . Thus this equation could have interaction between u_n and R . Indeed if one multiplies

the equation by u one finds

$$\begin{aligned} uu_t + u_x^2 + uu_{xx} &= 0 \\ \Rightarrow \quad 1/2 (u^2)_t + (u_x u)_x &= 0. \end{aligned} \tag{5.3}$$

b) Recursion Operators

The concept of rank can be used to restrict the possible form of a recursion operator for an equation - if it exists. Obviously in order to use the rank one must be able to define it for the equation being studied. The method assumes that all conservation laws are obtainable by multipliers - thus it will not necessarily work if a conservation law does not apply for all solutions of an equation (or indeed if the conjecture in chapter two is wrong and conservation laws exist for all solutions of an equation and yet no multipliers exist).

Magri (1978) shows that, for completely integrable equations, recursion operators arise from the connection between symmetries and conservation laws provided by symplectic and potential operators. The recursion operators that are usually studied provide relationships between the gradients of the conservation laws (see, for example, Magri (1978) and Lax (1976)) and thus they will provide a relationship between the multipliers of a completely integrable equation. (Recursion operators also exist which provide relationships between the conserved densities. For example, Gardner, Greene, Kruskal and Miura (1974)

gives such an operator for the conserved densities of the KdV equation.)

If one can define a rank for an equation then the multiplier will have terms which all belong to the same rank. Thus to go from one multiplier to another one will require an operator whose terms all belong to the same rank. If this were not the case then the terms in the operator could differ by an amount other than the subrank difference which defines the rank and thus the terms of the new multiplier would also have differences other than those allowed. For example, the Korteweg-de Vries equation

$$u_t + u_x u + u_{xxx} = 0 \quad (5.4)$$

has a subrank difference of $(1, -2, 0)$. Thus the most basic operator for the KdV equation will be of the form

$$R_0 = (d^2/dx^2 + au), \quad (5.5)$$

where a is constant.

If one considers the first two multipliers for the KdV equation,

$$u, u^2/2 + u_{xx} \quad (5.6)$$

one finds that for $a = 1/2$

$$u^2/2 + u_{xx} = (d^2/dx^2 + u/2)u. \quad (5.7)$$

However a quick calculation shows that R_0 does not produce the third multiplier from the second. Thus it is necessary to consider other forms of the recursion operator. Let M_{n+1} and M_n be multipliers of the KdV equation. If the basic form of the recursion operator, R_0 , provided a recursion operator for all

multipliers, one would have

$$M_{n+1} = R_0 M_n. \quad (5.8)$$

Since this is not the case, the recursion operator for the higher multipliers must be of the form,

$$R' M_{n+1} = R_1 M_n \quad (5.9)$$

where R_1 has the same rank as $R'R_0$. (R_1 is not necessarily equal to $R'R_0$.) The operator which produces the third multiplier of the KdV from the second (and in fact the $n+1$ th from the n th for all n) is (Magri (1978))

$$\frac{d}{dx} M_{n+1} = \left(\frac{d^3}{dx^3} + 2/3 u \frac{d}{dx} + 1/3 u_x \right) M_n. \quad (5.10)$$

Comparing this with (5.9) one sees that

$$R' = \frac{d}{dx},$$

and
$$R_1 = \frac{d^3}{dx^3} + 2/3 u \frac{d}{dx} + 1/3 u_x.$$

and, indeed, R_1 has the same rank as

$$R'R_0 = \frac{d^3}{dx^3} + u_x/2 + u/2 \frac{d}{dx}. \quad (5.11)$$

However, the coefficients are different. In fact R' and R_1 also take u to $u^2/2 + u_{xx}$ - so (R', R_1) forms the recursion operator for all n .

From this it can be seen that in order to find the recursion operators for higher order multipliers (or the general operator if it exists) one may need to study recursion operators other than the basic one obtained from the subrank difference of the equation.

The approach described above is not able to produce the

exact form of the recursion operators or to say whether a general recursion operator exists but it can provide a basic form of the operator which one could use in the search for a recursion operator.

c) The Henon-Heiles System

As was shown in chapter two for the non-linear water waves multipliers can be found for systems of partial differential equations. Not surprisingly, therefore, one can also find multipliers for systems of ordinary differential equations and the multipliers of such a system are studied in this section.

The Henon-Heiles system consists of the two ordinary differential equations in the two dependent variables X and Y

$$X_{tt} + AX + 2dXY = 0 \quad (5.12a)$$

$$Y_{tt} + BY - cY^2 + dX^2 = 0. \quad (5.12b)$$

For this system a subrank can be defined - the subrank (a, b, c) is the set of numbers (number of t-derivatives, number of factors of X, number of factors of Y.)

Multipliers for this system are (M_1, M_2) such that

$$(M_1 \times 5.12a) + (M_2 \times 5.12b) = d/dt T. \quad (5.13)$$

Thus interaction can take place between the two equations. Take M_1 and M_2 to be polynomials in terms with subrank up to (1, 4, 4) for one derivative and (2, 2, 2) for two derivatives (These are

the highest subranks for which calculations were done). If one multiplies the equations by these multipliers and reduces the product to its irreducible terms then one obtains the multipliers

$$\begin{aligned} M_1 &= F_1 X_t + F_2 Y_t + F_3 X_t Y + F_4 X Y_t \\ M_2 &= G_1 X_t + G_2 Y_t + G_5 X_t X \end{aligned} \quad (5.14)$$

and the following conditions on the coefficients

$$1. \quad cG_1 = -F_2 d, \quad F_2 A = G_1 B \quad (5.15)$$

$$2. \quad G_5 = 2d/c (F_3 - F_4) \quad (5.16)$$

$$3. \quad 2dF_1 + AF_3 + BG_5 - 2AF_4 - 2G_2 d = 0 \quad (5.17)$$

$$4. \quad F_2 = G_1 \quad (5.18)$$

$$5. \quad -F_3/2 = F_4 = G_5. \quad (5.19)$$

Assume $F_3 = F_4 = F_5 = 0$. Then the set of conditions reduces to

$$\begin{aligned} 2dF_1 - 2G_2 d &= 0 \\ \Rightarrow F_1 &= G_2 \end{aligned} \quad (5.20)$$

assuming non-zero d .

Thus the multipliers

$$\begin{aligned} M_1 &= X_t \\ M_2 &= Y_t \end{aligned} \quad (5.21)$$

produce an integral

$$1/2 (X_t^2 + AX^2 + Y_t^2 + BY^2) - c/3 Y^3 + dX^2 Y \quad (5.22)$$

for all A, B, c, d . This is the energy integral.

If G_1 and F_2 are not equal to zero then condition one implies

$$G_1/F_2 = -d/c = A/B. \quad (5.23)$$

But condition four implies $G_1 = F_2$. Thus another integral can be found if

$$-d/c = 1$$

and

$$(5.24)$$

$$A/B = 1.$$

For this set of values for A, B, c and d it is known that the Henon-Heiles system is separable (see, for example, Bountis et al (1982))

This multiplier then takes the form

$$(Y_t \times 5.12a) + (X_t \times 5.12b). \quad (5.25)$$

and the integral is

$$X_t Y_t + XY + 1/3 X^3 + XY^2. \quad (5.26)$$

Conditions two and five imply

$$G_5 = -6d/c G_5 \quad (5.27)$$

$$\Rightarrow (1 + 6d/c) G_5 = 0.$$

Thus $G_5 = 0$ (which implies, from condition five, that F_3 and F_4 equal zero) unless

$$d/c = -1/6. \quad (5.28)$$

Let $G_5 = 1$. Condition five implies that $F_4 = 1$ and $F_3 = -2$.

Condition three implies that

$$2d(F_1 - G_2) = -B + 4A \quad (5.29)$$

$$F_1 = -((B + 4A)/2d) + G_2.$$

Thus one appears to have an infinite number of possible multipliers of the form (it is assumed that $F_2 = G_1 = 0$)

$$\begin{aligned}
& (XY_t - 2X_tY + ((-B + 4A/2d) + G_2)X_t) \times 5.12a \\
& + (X_tX + G_2Y_t) \times 5.12b
\end{aligned} \tag{5.30}$$

where G_2 is arbitrary. However, (5.30) can be rewritten as

$$\begin{aligned}
& (XY_t - 2X_tY + (-B + 4A/2d)X_t) \times 5.12a \\
& + X_tX \times 5.12b + G_2((X_t \times 5.12a) + (Y_t \times 5.12b)).
\end{aligned} \tag{5.31}$$

The part of (5.28) involving G_2 is simply G_2 times the energy multiplier (5.17). Thus there is only one new multiplier,

$$\begin{aligned}
& (XY_t - 2X_tY + X_t(-B + 4A/2d)) \times 5.12a \\
& + X_tX \times 5.12b,
\end{aligned} \tag{5.32}$$

and, hence, two independent integrals for the case $d/c = -1/6$.

The two sets of values of A , B , c and d found above for which a second integral exists, $d/c = -1$, $A = B$ and $d/c = -1/6$ are two of the three sets of values for which the Henon-Heiles system is known to have a second integral. The third case is for

$$d/c = -1/16, B = 16A. \tag{5.33}$$

The integral for this case is not as simple as those found above. The integral is (Hall (1982))

$$\begin{aligned}
& 1/4 X_t^4 + (1/2 X^2 + 4X^2Y)X_t^2 - 4/3 X^3X_tY_t \\
& + 1/4 X^4 - 4/3 X^4Y - 8/9 X^6 - 16/3 X^4Y^2
\end{aligned} \tag{5.34}$$

and its multiplier is

$$\begin{aligned}
& (X_t^3 + X^2X_t + 8X^2YX_t - 4/3 X^3Y_t) \times 5.12a \\
& - (4/3 X^3X_t) \times 5.12b
\end{aligned} \tag{5.35}$$

Bountis et al (1982) show that the general solution of the Henon-Heiles system (with four arbitrary constants) has the Painleve property for only the three cases above. The strong

connection found between integrability and the Painlevé property suggests that these may be the only cases for which a second integral exists.

The first two cases are readily found using multipliers - if the third case were still unknown, could it also be easily found using multipliers? An initial quick search would find the first two cases but, given the complexity of the multiplier (a network multiplier, as indeed is the multiplier for the $d/c = -1/6$ case), the third case would probably not be found quickly. However a longer systematic search would reveal the third case.

The subrank differences for the Henon-Heiles system are $(2, 1, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, $(2, 0, 1)$, $(0, 0, 1)$, $(0, 0, 2)$ and $(0, 2, 0)$. Network multipliers could be constructed with these subrank differences using the method outlined in chapter four, section c. If one chooses (with a little foresight!) the basic subrank to be $(1, 2, 0)$, one gets a multiplier with the subranks $(1, 2, 0)$, $(3, 3, 0)$, $(1, 3, 0)$, $(1, 3, 1)$, $(3, 2, 1)$, $(1, 2, 1)$, $(1, 2, 2)$ and $(1, 4, 0)$. Since the multiplier (5.35) has subranks $(3, 3, 0)$, $(1, 3, 0)$, $(1, 3, 1)$ and $(1, 4, 0)$ the third multiplier would be found by using the basic subrank of $(1, 2, 0)$. Thus a simple network multiplier structure using a low order basic subrank produces the integral for the third case

Thus multipliers could have been used to find the three known cases of a second integral for the Henon-Heiles system and could certainly be used to search for higher order cases if their

existence is suspected.

As with any equation, the use of multipliers and the framework of chapter four cannot (at this stage of the technique's development at least) predict how many conservation laws (for partial differential equations) or integrals or constants of the motion (for ordinary differential equations) exist; the technique has its main use in the search for conservation laws and to find restrictions on the forms multipliers will take if they exist. Whether one finds all the integrals depends on the number that exist and the effort that is put into the search.

d) A Method for finding Conservation Laws

If one wishes to find the non-polynomial conservation laws of an equation or if the equation does not meet the various conditions found in the previous chapter then a systematic search method must be used.

In Chapter two a method of searching for polynomial multipliers (first presented in Suttie (1981)) was described. Although the method was, in theory, capable of dealing with high order multipliers, in practice the computer (a Burroughs B6700 using a FORTRAN program) could handle only the simplest multipliers before running out of storage space. Nevertheless, this method did show that computer algorithms could be developed

to search for the conservation laws of a partial differential equation. (Previous methods worked only for specific equations or types of equation. For example the method used by Kruskal et al (1970) works only for evolution equations.)

Choosing Multipliers

If nothing is known of the likely form of a multiplier (except, of course, its subrank differences which must be multiples of those in the equation) then one is confronted with the problem that there is an infinite number of possible multipliers. If a method is to be successful it must balance the necessity of reducing the problem to manageable proportions with the possibility of missing multipliers.

This can be done by making reasonable assumptions based on what is known about other equations and their conservation laws. For example, experience shows that any equation which has conservation laws has some multipliers which are of a simple form and equations which have an infinite number of conservation laws have multipliers which occur in some regular pattern. (In fact they often have recursion operators which produce the $n+1$ th multiplier from the n th - see section b) The KdV equation, for example, has multipliers as simple as u and $u^2 + 2u_{xx}$ and it has one and only one for every rank which contains one of u_{2n} , $n = 0, 1, 2, \dots$ (The only other ranks are those which contain u_{2n+1} . Thus every second rank provides a multiplier.)

It is reasonable therefore to assume that, if an equation

has no simple multipliers, then it probably has no multipliers, or if it has one or two simple ones but no others even of a fairly high order, then it probably does not have an infinite set of multipliers. (This assumption is only made for computational reasons - it is not a conjecture.)

In using the algorithm that is described in this chapter it is first necessary to restrict the general form of the multiplier to be tested. The total cardinalities of a multiplier must, by definition, all belong to the same superrank (or rank in the case of equations with only one subrank difference). However within a particular superrank there will be an infinite number of total cardinalities.

Two possible ways of restricting the number of total cardinalities are:

1. The number of subranks allowed in a multiplier can be restricted. In the equations studied so far the number of subranks in the simplest multipliers is comparable to the number of subranks in the equation. For example, for the equation [Abellanas and Galindo (1979)]

$$\begin{aligned} u_t - (2u^3 u_{xx} + 16u^2 u_x^2) + (3u^2 u_{xx} + 16uu_x^2) \\ - (uu_{xx} + 3u_x^2) = 0, \end{aligned} \quad (5.36)$$

one gets the multiplier

$$6u^5 - 15u^4 + 12u^3 - 3u^2. \quad (5.37)$$

This multiplier has four subranks compared to the equation's three.

It seems reasonable to assume that the simplest multipliers of all, or at least of the vast majority, of equations will have a similar property. Thus restricting the number of subranks to a value close to the number of subranks in the equation should find the lower order multipliers for an equation. These could then be used as a guide for the higher order multipliers. If one finds lower order multipliers one might also try to look for recursion operators using the method of section b).

2. The number of total cardinalities allowed for each subrank can be restricted. The most extreme way of doing this is to restrict the number to one (for example, to consider only polynomial multipliers). The number of results that have been found with polynomial multipliers suggests that such a restriction is not too severe. The multipliers involving x and t explicitly which were given for the Kadomtsev-Petviashvili (2.72) and KdV equations (2.51) also have a small number of total cardinalities for each subrank. Thus restrictions on the number of total cardinalities for each subrank may not greatly reduce the probability of finding multipliers. Of course one must be careful in one's choice of total cardinalities.

If one does need to place limits on the numbers of total cardinalities or subranks then the size of the limits will be determined by practical considerations such as the amount of computing space available.

One possible method of choosing the subranks of a multiplier

is the following:

Choose the first subrank. This could be done by starting with (1, 0, 0) then (1, 0, 1), (1, 1, 0), (1, 1, 1), (2, 0, 1), (2, 0, 1) et cetera. Once the first subrank has been chosen find all the combinations of n subranks (where n is the limit on the number of subranks) which have the same differences between subranks as the equation and which include the first subrank. For example, let $n = 3$. If (1, 0, 0) is the first subrank and if the equation differences are (1, 2, 1) and (3, 2, 2) then one gets the following combinations:

1. (1, 0, 0) (2, 2, 1) (3, 4, 2)
 2. (1, 0, 0) (2, 2, 1) (5, 4, 3)
 3. (1, 0, 0) (4, 2, 2) (5, 4, 3)
 4. (1, 0, 0) (4, 2, 2) (7, 4, 4).
- (5.38)

This method of producing combinations of subranks is not unique but the final set of combinations that is found will be the same no matter which method is used - only the order in which they are considered will change.

If the subranks are chosen in this way every subrank will appear as the first subrank. Thus if multipliers exist for which each term in the equation separately produces a divergence such multipliers will be found by this method. If one does not consider every subrank in this way (for example one may know of some restriction on the type of multiplier that can produce interaction) then this will not be the case and one will need to

consider such multipliers independently.

For each combination chosen, one must find all the possible total cardinalities within the limit chosen and then calculate all the possible terms in the multiplier. For example for combination (1) above, if one restricts one's attention to polynomial terms then the possible terms are

$$\begin{aligned}
 &u, u_{xxt}u, u_{xx}u_t, u_{xt}u_x, \\
 &u_{xxxxtt}u^2, u_{xxxxt}u_tu, u_{xxxx}u_{tt}u, u_{xxxx}u_t^2, \\
 &u_{xxx}u_{xtt}u_x, u_{xxx}u_{xt}u_t, u_{xxx}u_{xt}u_t, u_{xxx}u_{xt}u_t, u_{xxx}u_xu_{tt}, \\
 &u_{xxx}u_{xtt}u, u_{xxtt}u_{xx}u, u_{xxt}^2u, u_{xxt}u_{xx}u_t, u_{xx}^2u_{tt}, \\
 &u_{xxtt}u_x^2, u_{xxt}u_{xt}u_x, u_{xx}u_{xtt}u_x, u_{xx}u_{xt}^2.
 \end{aligned} \tag{5.39}$$

Thus the multiplier, if one exists, will be some linear combination of these terms.

It now remains to calculate the coefficients of this linear combination (if no multiplier exists the coefficients will all be zero). Multiply the equation by the polynomial consisting of the above set of terms with as yet undetermined coefficients. Reduce the product of equation and polynomial to its irreducible terms. The conditions on the coefficients of the polynomial for a divergence (and hence a conservation law) to occur are then found by equating the coefficients of the irreducible terms (which are sums containing the unknown coefficients) to zero.

The Method: Examples

The method described above is now illustrated by calculating the third multiplier for the KdV equation. Later the use of a

computer to find conservation laws is discussed and two results which were found using a program that incorporates this method are given.

Consider the non-t-derivative part of the KdV equation

$$u_x u + u_{xxx}. \quad (5.40)$$

Choose the first subrank to be (1, 4, 0). Since the difference between the equation terms is (1, -2, 0) one gets the combination (1, 4, 0), (2, 2, 0), (3, 0, 0) for the subranks of the multiplier. Thus the multiplier will be of the form

$$u_{xxxx} + au_{xx}u + bu_x^2 + cu^3. \quad (5.41)$$

Multiplying $u_x u + u_{xxx}$ by this multiplier one obtains

$$\begin{aligned} u_{xxxx}u_{xxx} + au_{xxx}u_{xx}u + bu_{xxx}u_x^2 + cu_{xxx}u^3 \\ + u_{xxxx}u_x u + au_{xx}u_x u^2 + bu_x^3 u + cu_x u^4. \end{aligned} \quad (5.42)$$

The terms $u_{xxxx}u_{xxx}$ and $u_x u^4$ equal divergences by themselves. The rest of the product has terms of two subranks;

(3, 5, 0)

$$au_{xxx}u_{xx}u + bu_{xxx}u_x^2 + u_{xxxx}u_x u. \quad (5.43)$$

and (4, 3, 0)

$$cu_{xxx}u^3 + au_{xx}u_x u^2 + bu_x^3 u. \quad (5.44)$$

Each of these two polynomials must equal a divergence if a conservation law is to be produced.

After performing the requisite manipulations one finds that the remaining irreducible terms are

$$(5/2 - 2b - a/2) u_{xx}^2 u_x \quad (5.45)$$

and $(-a + b - 3c)u_x^3 u. \quad (5.46)$

Thus the conditions on the coefficients for a divergence are

$$5 - 4b - a = 0 \quad (5.47)$$

and
$$b - a - 3c = 0. \quad (5.48)$$

Thus the polynomial

$$u_{xxxx} + (5-4b)u_{xx}u + bu_x^2 + (5-5b)/3 u^3, \quad (5.49)$$

where b is arbitrary, would be a multiplier for the ordinary differential equation,

$$u_x u + u_{xxx} = 0. \quad (5.50)$$

For a multiplier to exist for the KdV equation

$$(u_{xxxx} + au_{xx}u + bu_x^2 + cu^3)u_t \quad (5.51)$$

must also equal a divergence. Thus, since $u_{xxxx}u_t$ and u^3u_t are equal to divergences by themselves,

$$au_{xx}u_t + bu_x^2u_t \quad (5.52)$$

must equal a divergence. After manipulation the remainder is

$$(a - 2b)u_{xt}u_x. \quad (5.53)$$

Thus for a multiplier to exist for the KdV equation one has the conditions

$$5 - 4b - a = 0$$

$$a - 3c - b = 0 \quad (5.54)$$

$$a - 2b = 0.$$

There is only one solution to this set of conditions,

$$a = 5/3, b = 5/6, c = 5/18. \quad (5.55)$$

Thus the polynomial

$$u_{xxxx} + 5/3 u_{xx}u + 5/6 u_x^2 + 5/18 u^3 \quad (5.56)$$

will be a multiplier for the KdV equation. Comparing this with

the multipliers given in chapter 3, one sees that this is the third multiplier to within a multiplicative factor of 18/5.

A computer program was developed which systematically searched for multipliers for an equation using the method described in this chapter. The only data which needed to be input were details of the equation, an initial subrank for the multiplier and the limit on the number of subranks in a multiplier.

Initially several programs were produced which dealt with polynomial multipliers or with multipliers which have x and t dependence or which contain integrals of the form used in chapter four (see (4.10)). These were developed from the program used in Suttie (1981). The version dealing with integrals was tested with the KdV equation and the first two operators of McGuinness (1978) were readily found.

However these programs were not able to cope with sufficiently high order multipliers. Therefore another program was developed using the concepts of chapter four. The program calculated the irreducible terms that resulted from multiplying an equation by a multiplier and calculated the coefficients of the multipliers using the method described in this section. It was developed in PASCAL on a Burroughs B6900. The following two examples illustrate the type of results that were obtained using this program.

1. For the equation

$$u_t + u_x^2 + u_{xx} = 0. \quad (5.57)$$

Since this is an evolution equation the polynomial multipliers would involve only the x-derivatives. However, no polynomial multipliers were found for the combinations of three ranks involving the subranks (1, 1, 0) or (1, 2, 0). This is to be expected since the equation is of the form

$$u_t + u_{xx} + R = 0 \quad (5.58)$$

which was shown in chapter three to have no polynomial conservation laws. (The equation is not, itself, a conservation law.)

2. For the classical wave equation

$$u_{tt} - u_{xx} = 0. \quad (5.59)$$

Many multipliers were found by the computer for this equation.

Two such multipliers are

$$u_x^2 + u_t^2 \quad (5.60)$$

and

$$u_x u_{ttt} + u_{xt} u_{tt} + u_{xtt} u_t \quad (5.61)$$

which produce the conservation laws

$$\begin{aligned} (1/3 u_t^3 + u_t u_x^2)_t \\ - (1/3 u_x^3 + u_x u_t^2)_x = 0 \end{aligned} \quad (5.62)$$

and

$$\begin{aligned} (1/2 u_x u_{tt}^2 - u_{xx} u_x u_{tt} + u_{xxt} u_x u_t + 1/2 u_{xt}^2 u_x)_t \\ + (1/2 u_{tt}^2 u_t - u_{xtt} u_x u_t - 1/2 u_{xt}^2 u_t)_x = 0 \end{aligned} \quad (5.63)$$

respectively. Neither of the densities is a function of $u_t + u_x$.

Thus these conservation laws are additional to those mentioned in

chapter one, section e).

When using this program one must be careful to check that the multipliers are not trivial - the program cannot distinguish between the conditions for trivial and non-trivial multipliers. For example the program gives conditions which are satisfied by the multiplier

$$-3u_{ttt}u_{tt} + u_{xx}u_{ttt} + 2u_{xxt}u_{tt}. \quad (5.64)$$

This produces the conservation law

$$(-u_{tt}^3 + 2u_{tt}^2u_{xx} - u_{xx}^2u_{tt})_t = 0. \quad (5.65)$$

Using

$$u_{tt} = u_{xx} \quad (5.66)$$

one finds

$$(0)_t = 0. \quad (5.67)$$

Hence one has a trivial conservation law.

The examples given in this section show that the use of a computer is a good method of searching for the conservation laws of an equation.

Conclusion

In this thesis the multiplier has been studied as a new tool in the study of conservation laws - in particular, the conservation laws of completely integrable equations.

The usefulness of this tool arises from the important role played by conservation laws in physics, and especially in the area of 'soliton' physics. One of the first questions that needs to be asked in the study of conservation laws is

'What are the conservation laws of an equation?'

If a method can be found which can help to answer this question it is desirable if this method can also be used to study the properties of the conservation laws and if it provides insights into the connections between the conservation laws, the structure of the equation and other properties of the equation. The use of multipliers is such a method.

Multipliers have been used before both as a method of finding constants of the motion for ordinary differential equations and simple conservation laws for partial differential equations and in the general study of the properties of equations such as Noether's Theorem. However the use of multipliers has not previously been seen as a general method of study which could be used for a variety of equations and types of conservation law.

In chapter two it was shown that, for completely integrable equations, multipliers are identifiable as the gradients of the

conservation laws and using this identity they can be linked with many of the properties of such equations - for example symmetries and recursion operators.

Multipliers have previously been shown to exist for evolution equations and for a generalisation to higher order equations (Martínez Alonso (1979)) and it was conjectured in chapter two that multipliers will exist for all conservation laws which are valid for all solutions of an equation. Certainly it will require very unusual circumstances for a multiplier not to exist. For this to happen one must obtain (for an equation $G = 0$)

$$d_m J^m = FG + K$$

where F is a multiplier and K vanishes for solutions of the equation and is not linearly dependent on G (otherwise it could be incorporated into F). Since K vanishes for solutions to $G = 0$, the equation $K = 0$ must be true whenever $G = 0$. The possible existence of a K (and therefore the existence of a conservation law without a multiplier) cannot easily be ruled out but it does seem that it would occur only in very special circumstances.

If one already has the conserved densities for an equation then one can obtain the multipliers that produce these conservation laws. In chapter two, part c) the multipliers were found for the conservation laws of two simple ordinary differential equations and for a number of well-known and well-studied equations such as the Korteweg-de Vries, Sine-Gordon and Benjamin-Ono equations. It was shown that for the KdV and Sine-

Gordon equations the operators found by McGuinness (1978, 1980a,b) could be transformed into the multipliers for those solutions which were zero on the boundaries.

It is well-known that the variational derivative of a divergence is zero. Thus for an equation $G = 0$ and a multiplier F one will have

$$\frac{\delta}{\delta u} (FG) = 0.$$

Using a formula from Galindo (1979b) for the variational derivative of a product, a number of previously obtained results were obtained in a simpler way. It was also shown that for an equation of the form

$$u_t + u_n + R = 0, R \text{ polynomial,}$$

there exists at most one polynomial conservation law (the equation itself) unless n is odd. Thus the Diffusion equation

$$u_t + u_{xx} = 0$$

and Burger's Equation

$$u_t + u_x u + u_{xx} = 0$$

have no polynomial conservation laws other than the equations themselves. The multipliers of the equation $u_t + R = 0$ were shown to be also multipliers for the equation $u_t + u_x + R = 0$.

In chapter four a general framework was developed which enables a very wide range of equations to be studied using multipliers. Any equation for which a subrank can be defined can be studied in this way. The framework was developed by generalising the concepts of rank and irreducible terms first

used by Kruskal et al (1970).

For polynomial evolution equations that have a particular form (that is, the subrank difference in the non-t-derivative part of the equation must be the same) the form of the polynomial multipliers and therefore of the polynomial conservation laws was shown to be very restricted. This restriction was used to show that the equation

$$u_t + u^3 u_x + u_{xxx} = 0$$

does not have a conservation law for the fourth rank. Since the Korteweg-de Vries equation has a conservation law for every rank and since this equation is very similar in form to the Korteweg-de Vries equation one would expect that there would also be a conservation law for every rank for this equation as well (this view is strengthened by the required form of the multipliers which are very similar to those of the Korteweg-de Vries equation). Since there is no conservation law for the fourth rank it seems likely that this equation does not have an infinite number of polynomial conservation laws or if it does they would be of a totally different form to those of the Korteweg-de Vries equation (a possibility that contradicts the similarity of the forms of the multipliers). In fact it is known that this equation and indeed any equation of the form $u_t + u^p u_x + u_{2q+1} = 0, p \geq 2, q \geq 1$, has only three polynomial conservation laws (Miura (1974)).

Non-polynomial equations can be studied using the framework of chapter four but the form that the subranks take is not as

immediately obvious as for the polynomial equations. The Sine-Gordon equation can be given a subrank structure which has the same subrank difference as the Modified Korteweg-de Vries equation and the ranks for the Sine-Gordon correspond to those for the Modified Korteweg-de Vries equation. This agrees with the known relationship between the conservation laws of the two equations - the conservation laws of the Sine-Gordon equation are obtained from those of the Modified Korteweg-de Vries equation by the transformation $u \rightarrow \sqrt{3}/2 u$.

In section e) of chapter four it was shown that irreducible terms which are important in the use of the framework can be defined for a wide range of equations such as ordinary differential systems with two dependent variables and equations containing integrals of a particular type.

The use of this framework was illustrated in chapter five.

A generalisation of the earlier result for $u_t + u_n + R = 0$ was given which allows R to be non-polynomial - the generalised requirement is that R not have terms with the same subrank as u_n .

The basic form of a recursion operator for equations with only one rank was given in section b) of chapter five. The basic form has the same subrank difference as the equation. The closeness of the basic form of the recursion operator for the Korteweg-de Vries equation to the actual form of the operator shows that, by using the framework, one could simplify the search for recursion operators.

The Henon-Heiles system is an ordinary differential system of two equations in two dependent variables. Multipliers can be found for the integrals of this system and the use of irreducible terms allows the integrals to be found in a straight-forward way. The energy integral exists for this equation and a second integral exists for certain values of the equation constants - three sets of values are known. The first two of these sets were readily found using multipliers. It was shown how the third set could be obtained 'automatically' by a systematic search - no inspired guesswork, just a lot of paper should be required. Thus the use of multipliers can produce results for systems of this type - results which otherwise require detailed analysis such as a study of the Painlevé properties of the system.

For many equations no restrictions on the forms of the multipliers have been found and the only way to find multipliers is to systematically search for them. It is necessary to ensure that one's search method is practical (that is, it keeps the problem to a manageable size) and yet does not miss any possible multipliers. Such a method was given in the last section of chapter five. It has been incorporated in a computer program and the type of results obtainable by such a program were illustrated with two simple equations

$$u_t + u_x^2 + u_{xx} = 0$$

and the Classical Wave equation

$$u_{tt} + u_{xx} = 0.$$

The first was found to have no simple polynomial multipliers (as would be expected from the theorem in chapter three for equations of the form $u_t + u_n + R = 0$) while two new conservation laws were found for the Classical Wave equation - neither of whose densities is a function of $u_t + u_x$ (all functions of which are conserved densities for this equation).

The work described above shows that multipliers and the general framework that was developed in order to use them are useful in the study of conservation laws. Most importantly, they provide a method of answering the question

'What are the conservation laws of an equation?'

This method can be applied to any ordinary or partial differential equation and to other types of equation as well. Results can be found for equations which were previously obtained by a range of methods, most of which were useful for only one or a particular type of equation.

In addition to aiding the search for conservation laws, multipliers, or to be more precise, the general framework developed in chapter four, can be used to investigate the structure and properties of conservation laws and to relate aspects of conservation laws which other methods cannot relate. For example the existence of conservation laws for only particular ranks for the Korteweg-de Vries equation is related to the basic form of the recursion operator by the concept of subrank. One also sees that the reason Burger's equation does not

have polynomial conservation laws other than the equation itself is due, not to the nonlinearity as one might expect, but to the incompatibility of the terms u_t and u_{xx} - this incompatibility is also present in the linear Diffusion equation.

The work of this thesis provides a starting point - it sets out an analytical structure and illustrates how this may be used in many aspects of the study of conservation laws.

There are many possible directions that one could take from this starting point and the ideas presented in this thesis could evolve into a productive tool in the area of nonlinear physics. Possible directions include further development of the computer programs - perhaps using symbolic manipulation languages such as LISP - to allow analysis of systems of equations for which no results have yet been obtained. Network multipliers were only briefly considered here but a proper analysis of multi-equation systems would require the use of such multipliers and of the concept of superrank which was defined but only briefly used here. It may be possible to find restrictions on the forms of the network multipliers as was done for the multipliers of the Korteweg-de Vries equation. A computer may need to be used in this type of analysis.

An intriguing question is whether there is any significance in the observation that the Schrödinger operator used in the Inverse Scattering Transform for the solution of the Korteweg-de Vries equation,

$$L = \partial^2 / \partial x^2 + u$$

has the same subrank difference as the Korteweg-de Vries equation, its multipliers and its recursion operator!

The use of multipliers and irreducible terms and the concepts of subrank, rank and superrank are the basis of this thesis. They have proved to be capable of producing results for many equations and are able to provide insights into the structure of conservation laws that other methods cannot produce. There is promise of even more results and deeper insights if these basic concepts can be developed and used by others.

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